PART III: IONIZED NEBULAE
Section 7

Ionization equilibrium in H II Regions

As we have discussed previously, UV photons will photoionize neutral hydrogen atoms if they have with \( h\nu > 13.6 \text{eV} \) (= \( I_H \), the ground-state ionization potential for hydrogen). The excess energy \( h\nu - I_H \) is carried away as kinetic energy of the ejected photon (and hence goes into heating the gas).

The inverse process is recombination, and in equilibrium

\[
\text{H}^0 + h\nu \leftrightarrow \text{H}^+ + e^- 
\]

where essentially all ionizations are from the \( n = 1 \) level, but recombinations are to all levels. We will consider these processes in some detail, considering a pure hydrogen nebula.

7.1 Recombination

Recombination may occur to any level (principal quantum number \( n \)), and is followed by cascading down to the \( n = 1 \) level. The rate of recombinations per unit volume into level \( n \), \( \dot{n}_n \), depends on the density squared, and on the electron temperature:

\[
\dot{n}_n = n_e n_p \alpha_n(T_e) \simeq n_e^2 \alpha_n(T_e) \quad [\text{m}^{-3} \text{ s}^{-1}] 
\]

where \( \alpha_n(T_e) \) is the recombination coefficient (to level \( n \) at electron temperature \( T_e \)).

Obviously, the recombination rate is proportional to the number of available protons, \( n_p \), and the number of available electrons, \( n_e \); and for a pure hydrogen nebula, \( n_p = n_e \). The temperature dependence of \( \alpha_n(T_e) \) arises for two reasons:
(i) The higher the temperature the faster the electrons, and the more likely they are to encounter a proton; but, (ii) the faster the electron the less likely it is to be ‘captured’ by the proton. The latter term is more important, so the recombination coefficient is smaller at higher temperatures.

For the purpose of calculating the overall ionization balance, we evidently need the rate of recombinations to all levels (so called ‘case A recombination’):

\[ \dot{n}_A = \sum_{n=1}^{\infty} n_e n_p \alpha_n(T_e) \equiv n_e n_p \alpha_A(T_e) \]

However – recombination to \( n = 1 \) will always result in a photon with \( h\nu > 13.6 \text{eV} \) (i.e., a Lyman continuum photon), which will quickly be re-absorbed in a photoionization. Thus recombinations to \( n = 1 \) may be followed by ionizations from \( n = 1 \).

In the ‘on the spot’ approximation, photons generated in this way are assumed to be re-absorbed quickly and locally, and thus have no overall effect on the ionization balance.

In case B recombination, we therefore do not include recombination direct to the ground state;

\[ \dot{n}_B = \sum_{n=2}^{\infty} n_e n_p \alpha_n(T_e) \equiv n_e n_p \alpha_B(T_e). \] (7.2)

A reasonable numerical approximation to detailed calculations is

\[ \alpha_B(T_e) \simeq 2 \times 10^{-16} T_e^{-3/4} \text{ m}^3 \text{ s}^{-1}. \] (7.3)

### 7.2 Ionization

The lifetime of an excited level in the hydrogen atom \( (1/A_{ji}) \) is \( \sim 10^{-6} - 10^{-8} \text{s} \). This is very much less than the ionization timescale, so the probability of a photoionization from an excited state is negligible; essentially all ionization is from the ground state.

Suppose a star is embedded in a gas of uniform mass density. Then consider a volume element \( dV \) in a thin shell of thickness \( dr \) at distance \( r \) from the ionizing star.

Let the number of photons from the star crossing unit area of the shell per unit time be

\[ N_P(\nu) \text{ (m}^{-2} \text{s}^{-1}); \]

\[ ^1 \text{In principle, recombination of sufficiently energetic electrons to other levels can also result in ionizing photons, but this is a rare outcome in practice.} \]
then the photoionization rate from level 1 is

\[ \dot{n}_1 = \int_{0}^{\infty} a_\nu n(H^0) N_P(\nu) \, d\nu \]
\[ = \bar{a}_1 n(H^0) N_P(I) \quad (\text{m}^{-3} \, \text{s}^{-1}). \]  

(7.4)

where \( \nu_0 \) is the photon frequency corresponding to the ground-state ionization potential of hydrogen \((h\nu = 13.6 \text{ eV})\) \( N_P(I) \) is the number of ionizing photons, \( a_\nu \) is the ground-state photoionization cross-section (in units of \( \text{m}^2 \)), and \( \bar{a}_1 \) is the value of \( a_\nu \) averaged over frequency. In practice,

\[ a_\nu \simeq a_0 \left( \frac{\nu_0}{\nu} \right)^3 \]

for hydrogen (and hydrogen-like atoms), where \( a_0 \) is the cross-section at \( \nu_0 \). The photoionization cross-section therefore peaks at \( a_0 \); the flux of ionizing photons also typically peaks near \( \nu_0 \). It’s therefore not too inaccurate (and certainly convenient) to assume

\[ \bar{a}_1 \simeq a_0 = 6.8 \times 10^{-22} \text{ m}^2. \]

### 7.3 Ionization equilibrium

We first introduce \( x \), the degree of ionization, defined by

\[ n_p [= n(H^+)] = n_e = xn(H) \]

where \( n(H) \) is the total number of hydrogen nuclei, \( n_p + n(H^0) \); that is, \( x = 0 \) for a neutral gas, \( x = 1 \) for a fully ionized gas, and, in general,

\[ n(H^0) = (1 - x)n(H) \]

The condition of ionization equilibrium is that the number of recombinations equals the number of (ground-state) ionizations:

\[ \dot{n}_R(\equiv \dot{n}_B) = \dot{n}_1. \]

That is

\[ n_e n_p \alpha_B(T_e) = a_0(1 - x) n(H) N_P(I) \]

(from eqtns. (7.2) and (7.4)). Noting that, for our pure hydrogen nebula,

\[ n_e n_p = n_p^2 = x^2 n^2(H) \]
we obtain

\[
\frac{x^2}{(1-x)} = \frac{N_P(I)}{n(H)} \frac{a_0}{\alpha_B(T_e)}.
\]  \hspace{1cm} (7.5)

To estimate the photon flux \(N_P\) at a typical point in the nebula, we assume simple inverse-square dilution of the stellar flux [a good approximation except very near the star (where the geometry is slightly more complex) and near the edge of the nebula (where absorption becomes important\(^2\))]:

\[
N_P \simeq \frac{S_*}{4\pi r^2} \text{ [m}^{-2} \text{s}^{-1}] \hspace{1cm} (7.6)
\]

where \(S_*\) is the rate at which the star emits ionizing photons (s\(^{-1}\)). For representative numbers,

- \(n(H) = 10^8 \text{ m}^{-3}\)
- \(r = 1 \text{ pc} \times (3.08568025 \times 10^{16} \text{ m})\)
- \(S_* = 10^{49} \text{ s}^{-1}\)

(where \(S_*\) corresponds roughly to an O6.5 main-sequence star – similar to the ionizing star in the Orion nebula), eqtn. (7.6) gives

\[
N_P \simeq 8 \times 10^{14} \text{ m}^{-2} \text{s}^{-1}
\]

and eqtn. (7.5) becomes

\[
\frac{x^2}{(1-x)} \simeq 3 \times 10^4
\]  \hspace{1cm} (7.7)

(for \(T_e \sim 10^4\text{K}\)). We can solve this (e.g., by Newton-Raphson), giving \((1-x) = 3 \times 10^{-5}\); but we can see by simple inspection that \(x \simeq 1\) – i.e., where the gas is ionized it is, essentially, \textit{fully} ionized.

### 7.4 Nebular size and mass; the ‘Strömgren Sphere’

There is a simple physical limit to the size of a photoionized nebula; the total number of (case B) recombinations per unit time within a nebula must equal the total number of ionizing photons emitted by the star per unit time; that is, for a homogeneous nebula,

\[
\frac{4}{3} \pi R_S^3 n_e n_p \alpha_B(T_e) = S_* \hspace{1cm} (7.8)
\]  \hspace{1cm} \textit{\(^2\)Recall that the attenuation is \textit{exponential}, so there is a fairly rapid switch from ‘ionized’ to ‘neutral’ as the flux of ionizing photons rapidly diminishes.}
where $R_S$ is the (ionized) nebular radius, or Strömgren radius,

$$R_S = \left[ \frac{3}{4\pi} \frac{S_*}{n_e^2 \alpha_B(T_e)} \right]^{1/3}$$

(7.9)

where we’ve used the fact that $n_p = n_e$. The ionized volume is called the Strömgren sphere. Again adopting $S_* = 10^{49}$ s$^{-1}$ (and using $T_e \simeq 10^4$K in eqtn. (7.3) to evaluate $\alpha_B$) we find

$$R_S \simeq 7 \times 10^5 n_e^{-2/3} \text{ pc}$$

For typical densities, Strömgren radii are of order $\sim 10^0$–$10^2$ pc.

The mass is the volume times the (mass) density:

$$M_S = \frac{4}{3} \pi R_S^3 n_e m(\text{H})$$

$$= \frac{S_* m(\text{H})}{n_e \alpha_B(T_e)}$$

(from eqtn. (7.8)).
Section 8

Heating and Cooling

As for the diffuse neutral medium, the equilibrium gas temperature is determined by the balance between heating and cooling.

8.1 A pure hydrogen nebula

- **HEATING.** Photoionization of hydrogen heats the gas through the energy of freed electrons, $E = h\nu - I_H$ (where $I_H$ is the ionization potential of hydrogen, 13.6eV). We might suppose that, since both energetic photons and an abundant species are involved, this process should be more important for photoionized gas than in the neutral medium (where we recall that photoejection from dust dominates heating).

The energy input (per unit volume per unit time) is

$$G = \dot{n}_I Q (= \dot{n}_R Q \text{ in equilibrium}) \quad [\text{J m}^{-3} \text{s}^{-1}] \quad (8.1)$$

where $Q$ is the average energy input into the gas per photoionization ($= h\nu - I_H$).

- **COOLING.** On average, each recombination removes $\sim \frac{3}{2}kT_e$ (the average kinetic energy of the particles) of energy from the gas. The total loss per unit volume per unit time is therefore

$$L = \frac{3}{2}kT_e \dot{n}_R \quad [\text{J m}^{-3} \text{s}^{-1}] \quad (8.2)$$

In equilibrium, $L = G$ and hence

$$T_e = \frac{2Q}{3k}. \quad (8.3)$$

The value of $Q$ depends on the radiation field emitted from the star, and on the distance from the star (because of the $\nu^{-3}$ dependence of the photoionization cross-section, which means that
photons near the ionization edge are absorbed close to the star; hence the radiation field hardens with distance from the star).

As a rough approximation, we suppose the star radiates like a black body at temperature $T_*$, in which case the mean photon energy is obtained by dividing eqtn. (1.26) by (1.27),

\[ \overline{h\nu} = 2.7 kT_* \]

and, very roughly,

\[ Q \simeq \frac{3}{2} kT_* \quad (8.4) \]

(for $I_H \simeq kT_*$) whence, from eqtn. (8.3), we see that

\[ T_e \simeq T_* \]

Typical values for O-type stars are $T_* \simeq 30–50$ K; the implied electron temperatures in the nebula are much higher than observed values ($\sim 10^4$ K). To reconcile observed and computed temperatures, we need additional cooling processes.

For a pure hydrogen nebula, we have three possibilities:

- Free-free radiation (bremsstrahlung),
- Collisional excitation of hydrogen, and
- Collisional ionization of hydrogen.

However, none of these processes are important coolants in practice. It’s therefore necessary to relax the assumption of a pure hydrogen nebula.

### 8.2 Cooling by metal lines

As in the diffuse neutral medium, collisional excitation of forbidden lines of metals followed by radiative decay is an important coolant. We can see this directly in the spectra of photoionized nebulae; the total flux in forbidden lines of metals exceeds that in the hydrogen lines. The efficiency of collisional excitation is sufficiently efficient that it more than offsets the low abundances of metals compared to hydrogen.

We consider oxygen (one of the most important coolants) as an example. The IP of $O^0$, 13.6 eV, is almost the same as that for hydrogen, so where hydrogen is ionized, so is oxygen; while the IP of $O^+$ is 35.1 eV, so only the hotter O stars can produce $O^{2+}$.

All the labelled transitions in fig. 8.1 are forbidden (as electric dipole transitions; they can occur as magnetic dipole transitions or electric quadrupole transitions, with transition probabilities $A_{ji} \sim 1$ s$^{-1}$ [compare with allowed transitions, $A_{ji} \sim 10^8$ s$^{-1}$]). They are not
Figure 8.1: Simplified energy-level diagram for O\textsuperscript{+} (left) and O\textsuperscript{2+} (right). Unlabelled O\textsc{ii} and O\textsc{iii} wavelengths are in nm.

observed in the laboratory, where collisional de-excitation occurs before radiative decay can take place. However, at the much lower densities of nebulae, radiative decays can occur more rapidly than collisional de-excitations.

Since the oscillator strengths are small the probability of a photon from a forbidden transition being re-absorbed is small; these photons readily escape from (and hence cool) the nebula.

We designate the ground level of species \( I \), with number density \( n(I) \), as \( i \), and an excited upper level as \( j \); under nebular conditions \( n(I) \approx n_i(I) \). The collisional excitation rate from level \( i \) to \( j \) is

\[
\hat{n}_{ij} = n_e n_i(I) C_{ij}(T_e) \quad [\text{m}^{-3} \text{ s}^{-1}]
\]  

(8.5)

where \( C_{ij}(T_e) \) is the rate coefficient for collisional excitation (with typical values of order \( \sim 10^{-4} - 10^{-2} \text{ m}^3 \text{ s}^{-1} \)). The rate coefficient (and hence the collisional excitation rate) has a Boltzmann-like temperature dependence,

\[
C_{ij}(T_e) = \left( \frac{2\pi}{kT_e} \right)^{1/2} \frac{h}{4\pi^2 m_e^{3/2}} \frac{\Omega(ij)}{g_i} \exp \left\{ \frac{-\Delta E_{ij}}{kT_e} \right\}
\]

\[
\times \frac{1}{\sqrt{T_e}} \exp \left\{ \frac{-\Delta E_{ij}}{kT_e} \right\} \quad [\text{m}^3 \text{ s}^{-1}]
\]

(8.6)

(where \( \Omega(1, 2) \) is the so-called 'collision strength').
If each excitation is followed by radiative decay (and emission of a forbidden-line photon of energy $\Delta E_{ij}$), then the rate of energy loss is

$$L_{ij} = \dot{n}_{ij} \Delta E_{ij} = n_e n_i (I) C_{ij}(T_e) \Delta E_{ij} \quad [\text{J m}^{-3} \text{s}^{-1}]$$  \hspace{1cm} (8.7)

If we consider cooling due to collisional excitation to $^2D_{5/2}$ and $^2D_{3/2}$ in O$^+$ (the 372.9 and 372.6 nm lines, fig. 8.1), then

$$L(\text{O}^+) \simeq 1.1 \times 10^{-33} y(\text{O}^+) \frac{n^2(\text{H})}{\sqrt{T_e}} \exp \left\{ \frac{-3.89 \times 10^4}{kT_e} \right\} \quad \text{J m}^{-3} \text{s}^{-1}$$  \hspace{1cm} (8.8)

where $y(\text{O}^+)$ is the ionization fraction, $n(\text{O}^+)/n(\text{O})$, and we have used $n_e \simeq n(\text{H})$ & an oxygen abundance by number $n(\text{H})/n(\text{O}) = a(\text{O}) = 6 \times 10^{-4}$. Note the strong (exponential) dependence of the cooling rate on temperature.

As before (Section 5.3), to estimate an equilibrium temperature we set $G = L$; from eqtns. (8.1) and (8.4) we have

$$G(= \dot{n}_I Q) = \dot{n}_R Q \simeq n^2(\text{H}) \alpha_B(T_e) kT_e$$

and using eqtn. (7.3) for $\alpha_B$ we find, from eqtn. (8.8),

$$T_e^{1/4} \exp \left\{ \frac{-3.89 \times 10^4}{kT_e} \right\} = 3.75 \times 10^{-6} T_*$$

Numerically,

<table>
<thead>
<tr>
<th>Stellar Temp.</th>
<th>Equilbm. Nebular</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_e$ (K)</td>
<td>Temp. $T_e$ (K)</td>
</tr>
<tr>
<td>2 $\times$ 10$^4$</td>
<td>7500</td>
</tr>
<tr>
<td>4 $\times$ 10$^4$</td>
<td>8600</td>
</tr>
<tr>
<td>6 $\times$ 10$^4$</td>
<td>9300</td>
</tr>
</tbody>
</table>

which is in satisfactory agreement with observations.

Note that over the entire factor-3 range of relevant stellar temperatures (cooler stars don’t produce ionized nebulae; hotter ‘normal’ stars aren’t found), the nebular temperature only varies by $\sim 20\%$. Why? First, as $T_e$ increases, the recombination rate decreases, and so the number density of neutral hydrogen goes down. A decrease in neutral hydrogen number density reduces the rate of heating. Secondly, as $T_e$ increases, the cooling rate, eqtn. (8.6), goes up. Both effects oppose the trend to increasing $T_e$. This feedback mechanism, or ‘thermostat’ regulates the temperatures of $\text{H II}$ regions.

### 8.3 Thermalization in the gas

A pure hydrogen nebula contains neutral hydrogen atoms, protons, and electrons.
Photoionization continuously injects energy into the nebula, through the energy of the photoelectrons, which is determined by the energy distribution of ionizing photons $(E = h\nu - \text{I.P.})$ – that is, by the effective temperature of the ionizing star(s).

Following photoejection of the electron, collisions redistribute the electron energy among all particles, thereby increasing the kinetic temperature. (The ‘collisions’ are not physical impacts, of course, but Coulomb interactions.) This happens so quickly (compared to ionization/recombination timescales) that it can be regarded as essentially instantaneous, but the energy redistribution is nonetheless hierarchical:

- The ejected electrons first share their energy with other electrons
- The electrons transfer energy to the protons, until equipartition of energy is achieved. (This is a slower process because the mass difference between electrons and protons makes the energy transfer inefficient.)
- Finally, the neutrals (which are less affected by coulomb interactions) gain energy
Free-free emission (or \textit{bremstrahlung}) is generated by the deceleration of thermal electrons in the electric field of ions. This is a continuous process in wavelength, but the emission is most readily observed in the radio regime, where it dominates the emission from an ionized gas. Here we will discuss its application to ionized nebulae.

Self-absorption of free-free emission within the nebula can be significant, and must be taken into account – i.e., we must consider the radiative transfer within the nebula. To do this we recall definitions from Section 1

- $I_\nu$, the (‘specific’) intensity of radiation – the rate of energy flow energy,
  - per unit frequency,
  - per unit area,
  - per unit solid angle,
  - per unit time.
  SI Units are thus $\text{J Hz}^{-1} \text{m}^{-2} \text{sr}^{-1} \text{s}^{-1} (= \text{J m}^{-2} \text{sr}^{-1})$

- $j_\nu$, the \textit{emission coefficient} – the radiant energy emitted by the gas,
  - per unit frequency,
  - per unit volume,
  - per unit solid angle,
  - per unit time.
  Units are thus $\text{J Hz}^{-1} \text{m}^{-3} \text{sr}^{-1} \text{s}^{-1} (= \text{J m}^{-3} \text{sr}^{-1})$

- $k_\nu$, the \textit{absorption coefficient}, or opacity per unit volume.

From Section 3.1, the equation of radiative transfer can be written as

$$\frac{dI_\nu}{ds} = j_\nu - k_\nu I_\nu,$$  \hspace{1cm} (3.1)
where the ratio $j_\nu/k_\nu$ is the Source Function, $S_\nu$. For systems in thermodynamic equilibrium, the source function is given by the Planck function, $B_\nu$, and $j_\nu$ and $k_\nu$ are related through the Kirchhoff relation,

$$j_\nu = k_\nu B_\nu(T); \quad \text{i.e., } S_\nu = B_\nu$$

(Section 10.1). Because free-free radiation is an essentially collisional process, in this respect the nebula is in thermodynamic equilibrium, and we can use $S_\nu = B_\nu$. Then eqtn. (3.1) can be written in the form

$$\frac{dI_\nu}{d\tau_\nu} = B_\nu(T_e) - I_\nu$$

(cp. eqtn. (3.2)), where we have use the definition of optical depth,

$$d\tau_\nu = k_\nu ds.$$

The solution of this first-order differential equation is

$$I_\nu = B_\nu(T_e)(1 - \exp\{-\tau_\nu\}) \quad \text{(9.1)}$$

where $\tau_\nu$ is the total optical depth through the region. Note that we have made two implicit assumptions –

- $T_e$ is constant throughout the region
- $I_\nu = 0$ at $\tau_\nu = 0$; i.e., there is no external or background radiation.

There are two obvious limiting forms of eqtn. (9.1):

1. For $\tau_\nu \ll 1$, $\exp\{-\tau_\nu\} \to [1 - \tau_\nu]$ and

$$I_\nu \simeq B_\nu(T_e)\tau_\nu \quad \text{(9.2)}$$

2. For $\tau_\nu \gg 1$, $\exp\{-\tau_\nu\} \to 0$, and

$$I_\nu \simeq B_\nu(T_e) \quad \text{(9.3)}$$

**Free-free (reference only)**

From Allen (AQ), the free-free opacity is given by:

$$k_{ff} = \frac{4\pi Z^2 e^6 g_f}{3\sqrt{3} \hbar c m_e^2 v^2 n_e n_i} \left[ \text{m}^{-1} \right]$$
where $g_{ff}$ is the free-free gaunt factor. Using

$$v_e = \sqrt{\frac{\pi kT}{2m_e}}$$

as the mean velocity, and allowing for stimulated emission,

$$k_\nu = 3.692 \times 10^8 Z^2 g_{ff} n_e n_i \left( \frac{\nu}{\sqrt{T}} \right) \left[ 1 - \exp \left\{ \frac{-h\nu}{kT} \right\} \right]$$

for electron velocity $v_e$, ionic charge $Z$, electron and ion densities $n_e n_i$, and cgs units throughout ($\lambda$ in cm). Normally $n_e \simeq n_i \simeq n(H)$ (and certainly $n_e, n_i \propto n(H)$); then expanding the exponential term,

$$\exp \left\{ \frac{-h\nu}{kT} \right\} \simeq 1 - \frac{h\nu}{kT} + \frac{1}{2} \left( \frac{h\nu}{kT} \right)^2 \ldots$$

gives

$$k_{ff} \propto g_{ff} \nu^{-2} T^{-3/2} n^2(H)$$

In the case of free-free radiation at radio wavelengths, the opacity can be approximated by

$$k_\nu \propto \nu^{-2.1} T_e^{-1.35} n_e n_p$$

(where we have made allowance for the weak $\nu, T$ dependences of $g_{ff}$), so that for a pure-hydrogen nebula ($n_e = n_p$)

$$\tau_\nu \propto \nu^{-2.1} T_e^{-1.35} n_e^2 L$$

(9.4)

for path length $L$ through the nebula; the quantity $n_e^2 L$ is a fixed quantity, called the Emission Measure (usually expressed in units of m$^{-6}$ pc), for a given nebula. Note that the optical depth is smaller at higher frequencies (shorter wavelengths).

At radio frequencies we can use the Rayleigh-Jeans approximation to the Planck function (eqtn. 1.23),

$$B_\nu(T_e) = \frac{2h\nu^3}{c^2} \frac{1}{\exp\{h\nu/kT_e\}} \simeq \frac{2kT_e}{c^2} \nu^2$$

(9.5)

[since $(h\nu)/(kT_e) \ll 1$ for small $\nu$, so $\exp\{(h\nu)/(kT_e)\} \simeq 1 + (h\nu)/(kT_e)$].

$$\log I_\nu \quad \log \nu$$

$\nu^2$ $\nu^{-0.1}$
In the optically thin limit (small optical depth; high frequencies, short wavelengths) eqtns. (9.4), (9.5) and (9.2) give

\[ I_\nu \propto \nu^{-0.1}T_e^{-0.35}n_e^2L \]

while in the optically thick limit (large optical depths; small frequencies, long wavelengths) eqtns. (9.4), (9.5) and (9.3) give give

\[ I_\nu \propto \nu^2T_e. \]

That is, we can determine the nebular temperature, directly (and independent of distance\(^1\) from the intensity at optically thick frequencies; then knowing \(T_e\), we can determine the emission measure from the emission at any optically thin frequency (or from eqtn. (9.4) by determining the ‘turnover frequency’, \(\nu_0\), at which the optical depth is unity).

\(^1\)For a spatially resolved source; cf. section 1.3