Section 11

Timescales

11.1 Radiation transport in stars

Deep inside stars the radiation field is very close to black body. For a black-body distribution the photon number density at temperature $T$ is given by

$$n = 2 \times 10^7 T^3 \quad [\text{m}^{-3}]$$

so the average photon energy is

$$\bar{E} = U/n \simeq 4 \times 10^{-23} T \quad [\text{J photon}^{-1}]$$ (11.2)

(from eqtn. 7.26).

The core temperature of the Sun is $T_c \simeq 1.5 \times 10^7$ K, whence $\bar{E} = 3.5$ keV (i.e., photons in the X-ray regime).

Light escaping the surface of the Sun ($T_{\text{eff}} \simeq 5770$K) has a mean photon energy $\sim 3 \times 10^3$ smaller, in the optical.

The source of this degradation in the mean energy is the coupling between radiation and matter. Photons obviously don’t flow directly out from the core, but rather they diffuse through the star, travelling a distance of order the local mean free path, $\ell$, before being absorbed and re-emitted in some other direction (a ‘random walk’). The mean free path depends on the opacity of the gas:

$$\ell = 1/(\kappa_m \rho)$$ (11.3)

where $\kappa_m$ is the mass opacity$^1$, with units m$^2$ kg$^{-1}$.

$^1$Opacity can also be expressed in length units
After $n_{\text{sc}}$ scatterings the distance travelled is, on average (it’s a statistical process), $\sqrt{n_{\text{sc}} \ell}$. Thus to travel a distance $R_\odot$ we require

$$n_{\text{sc}} = \left( \frac{R_\odot}{\ell} \right)^2.$$  \hspace{1cm} (11.4)

Solar-structure models give an average mean free path $\ell \simeq 1$ mm (justifying LTE!); with $R_\odot \simeq 7 \times 10^8$ m,

$$n_{\text{sc}} \simeq 5 \times 10^{23}$$

The total distance travelled by a (fictitious!) photon travelling from the centre to the surface is $n_{\text{sc}} \times \ell \simeq 5 \times 10^{20}$ m ($\sim 10^{12} R_\odot$!), and the time to diffuse to the surface is $(n_{\text{sc}} \times \ell)/c \simeq 5 \times 10^4$ yr.

[More detailed calculations give $17 \times 10^4$ yr; why? Naturally, there are regions within the Sun that have greater or lesser opacity than the average value, with the largest opacities in the central $\sim 0.4 R_\odot$ and in the region immediately below the photosphere. Because of the ‘square root’ nature of the diffusion, a region with twice the opacity takes four times longer to pass through, while a region with half the opacity takes only $1.414$ times shorter; so any non-uniformity in the opacity inevitably leads to a longer total diffusion time.]

### 11.2 Stellar timescales

A number of additional characteristic timescales can be established:

#### 11.2.1 Dynamical timescale

‘Hydrostatic equilibrium’ approach

If we look at the Sun in detail, we see that there is vigorous convection in the envelope. With gas moving around, is the assumption of hydrostatic equilibrium justified? To address this question, we need to know how quickly displacements are restored; if this happens quickly (compared to the displacement timescales), then hydrostatic equilibrium remains a reasonable approximation even under dynamical conditions.

We can write an equation of motion,

$$\rho a = \rho g + \frac{dP}{dr}$$  \hspace{1cm} (11.5)
where $g$ is the acceleration due to gravity and

$$a = \frac{d^2r}{dt^2}$$

is the net acceleration. As the limiting case we can 'take away' gas-pressure support (i.e., set $dP/dr = 0$), so our equation of motion becomes

$$\frac{d^2r}{dt^2} = -\frac{Gm(r)}{r^2}.$$

Integrating,

$$r = \frac{1}{2}gt^2 \quad \text{(for } v_0 = 0)$$

(11.6)

but $g = Gm(r)/r^2$, so, identifying the time $t$ in eqtn. 11.6 with a dynamical timescale, we have

$$t_{\text{dyn}} = \sqrt{\frac{2r^3}{Gm(r)}}. \quad \text{(11.7)}$$

Departures from hydrostatic equilibrium are restored on this timescale (by gravity in the case of expansion, or gas pressure in the case of contraction). In the case of the Sun,

$$t_{\text{dyn}} = \sqrt{\frac{2R^3}{GM_{\odot}}} \approx 37 \text{ min.}$$

(If you removed gas-pressure support from the Sun, this is how long it would take a particle at the surface to free-fall to the centre.)

‘Virial’ approach

The ‘hydrostatic equilibrium’ approach establishes a collapse timescale; as an alternative, we can consider a pressure-support timescale. Noting that a pressure wave propagates at the sound speed, this dynamical timescale can be equated to a sound-crossing time.

The sound speed is given by

$$c_s^2 = \gamma \left( \frac{kT}{\mu m(H)} \right)$$

(11.8)

(where $\gamma = C_p/C_v$, the ratio of specific heats at constant pressure and constant volume).

From the virial theorem,

$$2U + \Omega = 0$$

(10.21)
with
\[ U = \int_{V} \frac{3}{2} kT n(r) \, dV \]  \hspace{1cm} (10.16)
\[ = \int_{V} \frac{3}{2} kT \frac{\rho}{\mu m(H)} r^2 \, dV \]  \hspace{1cm} (11.9)

and
\[ \Omega = - \int_{M}^{0} \frac{Gm(r)}{r} \, dm(r) \]  \hspace{1cm} (10.20)
\[ = - \int_{V} \frac{Gm(r)}{r} \rho(r) \, dV \]  \hspace{1cm} (11.10)

From eqtns. 10.21, 11.9 and 11.10
\[ \frac{3kT}{\mu m(H)} = \frac{Gm(r)}{r} \]

that is, from eqtn. 11.8,
\[ \frac{3}{\gamma c_s^2} = \frac{Gm(r)}{r} \]

For a monatomic gas $\gamma = \frac{5}{3}$, giving
\[ c_s^2 = \frac{5}{9} \frac{Gm(r)}{r} \]

and the sound crossing time is
\[ t = \frac{r}{c_s} = \sqrt{\frac{9/5r^3}{Gm(r)}} \]  \hspace{1cm} (11.11)

(which is within $\sim 10\%$ of eqtn. 11.7).

### 11.2.2 Kelvin-Helmholtz/Thermal Timescales

Before nuclear fusion was understood, gravitational contraction was considered as a possible source of the Sun’s luminosity.\(^2\) The time over which the Sun’s luminosity can be powered by this mechanism is the *Kelvin-Helmholtz* timescale.

\(^2\)Recall from Section 10.7 that half the gravitational potential energy lost in contraction is radiated away, with the remainder going into heating the star.
The available gravitational potential energy is
\[ \Omega = \int_0^R \frac{-Gm(r)}{r} \, dm(r) \quad (10.20) \]
but
\[ m(r) = \frac{4}{3} \pi r^3 \bar{\rho} \quad \text{so} \quad dm(r) = 4\pi r^2 \bar{\rho} \, dr \]
and
\[ \Omega = \int_0^R -G \frac{16\pi^2}{3} r^4 \bar{\rho}^2(r) \, dr \\
\simeq \frac{16}{15} \pi^2 G \bar{\rho}^2 R^5 \quad (11.12) \]
(assuming \( \bar{\rho}(r) = \bar{\rho}(R) \)).

The Kelvin-Helmholtz timescale is therefore
\[ t_{KH} = \frac{|\Omega(\odot)|}{L_\odot}. \quad (11.13) \]
which for \( \bar{\rho} = 1.4 \times 10^3 \, \text{kg m}^{-3}, \Omega = 2.2 \times 10^{41} \, \text{J} \) is \( t_{KH} \simeq 10^7 \, \text{yr}. \)

The Kelvin-Helmholtz timescale is often identified with the \textit{thermal timescale}, but the latter is more properly defined as
\[ t_{Th} = \frac{U(\odot)}{L_\odot}, \quad (11.14) \]
which (from the virial theorem) is \( \sim \frac{1}{2} t_{KH} \).

\subsection*{11.2.3 Nuclear timescale}

We now know that the source of the Sun’s energy is nuclear fusion, and we can calculate a corresponding nuclear timescale,
\[ t_N = \frac{f M c^2}{L} \quad (11.15) \]

\footnote{At the time that this estimate was made, the fossil record already indicated a much older Earth (\( \sim 10^9 \, \text{yr} \)). Kelvin noted this discrepancy, but instead of rejecting contraction as the source of the Sun’s energy, he instead chose to reject the notion of evolution.}
where $f$ is just the fraction of the rest mass available to the relevant nuclear process. In the case of hydrogen burning, expressed as a fraction this ‘mass defect’ is 0.007, so we might expect

$$t_N = \frac{0.007M_\odot c^2}{L_\odot} \simeq 10^{11} \text{ yr}.$$  

However, in practice, only the core of the Sun – about $\sim 10\%$ of its mass – takes part in hydrogen burning, so its nuclear timescale for hydrogen burning is $\sim 10^{10}$ yr. Other evolutionary stages have their respective (shorter) timescales.