2. A major weakness of Λ iteration is that it is a local method; that is, it takes no account of the effect that the correction $\delta T(\tau)$ at depth τ has on the mean intensity $J_\nu(\tau')$ in some adjacent layer at depth τ'. A better method would be sensitive to the *global* status of the atmosphere; that is, we would prefer to use a correction in an integral form.

3. Furthermore, because radiative equilibrium is enforced *locally* (rather than globally), the method can lead to $\mathrm{d}F/\mathrm{d}\tau = 0$, as required, but not at the desired value for the flux F_0. More generally, the method can stabilize at an incorrect solution – a very bad numerical property.

As a result of these limitations, Λ iteration is not used as a practical solution method – although often wheeled out as a simple demonstration of how to solve an atmospheric structure in principle, it is also a good example of how *not* to do it in practice. However, towards the end of the 20th century, it was realised that its convergence properties could be dramatically improved through the introduction of modified, 'accelerated' or 'approximate', lambda operators. These so-called ALI methods are now a mainstay of modern stellar-atmosphere computer programs.

Nevertheless, other techniques remain in use, and are (arguably) easier to demonstrate, including (e.g.) Avrett-Krook and Unsöld–Lucy iteration; we'll review the latter as a practical alternative to simple Λ iteration.

9.3 Unsöld–Lucy iteration

The Unsöld–Lucy method incorporates constraints on both the absolute value of the flux and its (lack of) depth dependence. With minor modifications, this method is embodied in the 'state of the art' modelling code `PHOENIX`.

Although we want to relax the grey-atmosphere approximation, it's still convenient to avoid the full frequency dependence of opacities by defining several frequency-integrated, flux-weighted forms (where all quantities with subscripts are to be understood to be functions of [optical] depth in the atmosphere). This is tolerable because at this stage we're not really concerned with the frequency-dependent spectrum, but just the overall radiative energy transport. We define the following:

$$\text{Planck mean opacity:} \qquad k_\mathrm{P} = \frac{\int_0^\infty k_\nu^\mathrm{a} B_\nu \,\mathrm{d}\nu}{\int_0^\infty B_\nu \,\mathrm{d}\nu}, \qquad \equiv \frac{\int_0^\infty k_\nu^\mathrm{a} B_\nu \,\mathrm{d}\nu}{B};$$

$$\text{Eddington-flux mean opacity:} \qquad k_\mathrm{H} = \frac{\int_0^\infty k_\nu H_\nu \,\mathrm{d}\nu}{\int_0^\infty H_\nu \,\mathrm{d}\nu}, \qquad \equiv \frac{\int_0^\infty k_\nu H_\nu \,\mathrm{d}\nu}{H};$$

$$\text{intensity mean opacity:} \qquad k_\mathrm{J} = \frac{\int_0^\infty k_\nu^\mathrm{a} J_\nu \,\mathrm{d}\nu}{\int_0^\infty J_\nu \,\mathrm{d}\nu}, \qquad \equiv \frac{\int_0^\infty k_\nu^\mathrm{a} J_\nu \,\mathrm{d}\nu}{J},$$

with corresponding (frequency-independent) optical-depth increments $\mathrm{d}\tau_\mathrm{P}$, $\mathrm{d}\tau_\mathrm{H}$, and $\mathrm{d}\tau_\mathrm{J}$ (= $-k_\mathrm{P}\,\mathrm{d}r$, $-k_\mathrm{H}\,\mathrm{d}r$, and $-k_\mathrm{J}\,\mathrm{d}r$), where $k_\nu = k^\mathrm{a} + k_\nu^\mathrm{s}$ is the sum of 'true' and scattering opacities (cf. eqtn. 7.11).

9.3.1 Zeroth moment

Starting from the zeroth moment of the transfer equation (integrating the transfer equation over solid angle, as in our discussion of Milne's first equation; Section 8.2.1) we previously obtained the flux derivative in the form

$$\frac{\mathrm{d}H_\nu(\tau_\nu)}{\mathrm{d}r} = k_\nu S_\nu(\tau_\nu) - k_\nu J_\nu(\tau_\nu), \tag{8.4}$$

(recalling that the Eddington flux is $H_\nu = F_\nu/4\pi$). Using eqtn. (7.11) for S_ν and integrating over frequency, then at some (frequency-independent) depth τ_P in the atmosphere

$$\int_0^\infty \frac{\mathrm{d}H_\nu(\tau_\mathrm{P})}{\mathrm{d}r}\,\mathrm{d}\nu = \int_0^\infty k_\nu^\mathrm{a} B_\nu\left(T(\tau_\mathrm{P})\right)\,\mathrm{d}\nu - \int_0^\infty k_\nu^\mathrm{a} J_\nu(\tau_\mathrm{P})\,\mathrm{d}\nu.$$

Using the intensity- and Planck-mean volume opacities on the right-hand side this becomes

$$\frac{\mathrm{d}H(\tau_\mathrm{P})}{\mathrm{d}r} = k_\mathrm{P} B\left(T(\tau_\mathrm{P})\right) - k_\mathrm{J} J(\tau_\mathrm{P}).$$

Dividing both sides by k_P and rearranging we obtain

$$B\left(T(\tau_\mathrm{P})\right) = \frac{k_\mathrm{J}}{k_\mathrm{P}} J(\tau_\mathrm{P}) - \frac{\mathrm{d}H(\tau_\mathrm{P})}{\mathrm{d}\tau_\mathrm{P}}. \tag{9.3}$$

In principle, eqtn. (9.3) allows us to compute the frequency-integrated Planck source function at depth τ_P; or, essentially equivalently, the temperature structure temperature $T(\tau_\mathrm{P})$. It has the desireable property that all the terms are frequency-averaged or frequency-integrated (so we don't have to explicitly evaluate them frequency by frequency).

However, to use eqtn. (9.3) in practice, knowing that we want a radiative-equilibrium temperature structure in which the radiative flux H is constant with depth,[6] we see that we require a useful expression for $J(\tau_\mathrm{P})$. To achieve that, we look to the first moment.

9.3.2 First moment

From the first moment of the transfer equation we saw that

$$\frac{\mathrm{d}K_\nu(\tau_\nu)}{\mathrm{d}\tau_\nu} = \frac{F_\nu(\tau_\nu)}{4\pi} \equiv H_\nu(\tau_\nu); \tag{8.7}$$

but in the Eddington (two-stream) approximation $K_\nu = J_\nu/3$ (eqtn. 8.16), and so

$$\frac{\mathrm{d}J_\nu(\tau_\nu)}{\mathrm{d}\tau_\nu} = 3H_\nu(\tau_\nu).$$

[6] Hence the $\mathrm{d}H(\tau_\mathrm{P})/\mathrm{d}\tau_\mathrm{P}$ term goes to zero when we achieve a solution with a correct, radiative-equilibrium, temperature structure. However, we can't ignore the term at this stage, because it will be non-zero for any incorrect 'first guess' temperature structure.

Using $\mathrm{d}\tau_\nu = -k_\nu \mathrm{d}r$, and integrating over frequency, then at some Planck-mean optical depth τ_P

$$\int_0^\infty \frac{\mathrm{d}J_\nu(\tau_\mathrm{P})}{\mathrm{d}r}\,\mathrm{d}\nu = -3\int_0^\infty k_\nu H_\nu(\tau_\mathrm{P})\,\mathrm{d}\nu,$$

or, using the flux-mean opacity on the right-hand side

$$\frac{\mathrm{d}J(\tau_\mathrm{P})}{\mathrm{d}r} = -3k_\mathrm{H}H(\tau_\mathrm{P}). \tag{9.4}$$

Dividing both sides of eqtn. (9.4) by $-k_\mathrm{P}$ and integrating over optical depth leads to

$$J(\tau_\mathrm{P}) = \int_0^{\tau_\mathrm{P}} 3\frac{k_\mathrm{H}}{k_\mathrm{P}}H(t)\,\mathrm{d}t + J(0)$$

where $J(0)$ is a constant of integration corresponding to the mean intensity at the surface. Recalling that

$$J(0) = F(0)/2\pi, = 2H(0) \tag{8.17}$$

in the Eddington two-stream approximation, we obtain

$$J(\tau_\mathrm{P}) = \int_0^{\tau_\mathrm{P}} 3\frac{k_\mathrm{H}}{k_\mathrm{P}}H(t)\,\mathrm{d}t + 2H(0). \tag{9.5}$$

This is our required expression for $J(\tau_\mathrm{P})$, the mean intensity at some depth τ_P in the atmosphere.

9.3.3 The correction

Combining eqtns. (9.3) and (9.5) gives, in effect, the temperature as a function of Planck mean optical depth, in terms of the Eddington flux – which, for given T_eff, we know.

$$B\,(T(\tau_\mathrm{P})) = \frac{k_\mathrm{J}}{k_\mathrm{P}}\left[\int_0^{\tau_\mathrm{P}} 3\frac{k_\mathrm{H}}{k_\mathrm{P}}H(t)\,\mathrm{d}t + 2H(0)\right] - \frac{\mathrm{d}H(\tau_\mathrm{P})}{\mathrm{d}\tau_\mathrm{P}}, \qquad = \sigma T^4(\tau_\mathrm{P})/\pi. \tag{9.6}$$

As in our outline of Λ iteration, some initial trial solution for the temperature structure $T_1(\tau_\mathrm{P})$ will, in general, predict an initial set of fluxes $H_1(\tau_\mathrm{P})$ that vary with depth; while in radiative equilibrium the correct solution should give constant flux H for all τ_P.

We therefore need to evaluate the correction required to the temperature – or, equivalently, the correction to frequency-integrated Planck function, $B\,(T(\tau_\mathrm{P}))$ in eqtn. (9.6), which we can translate directly into a temperature correction. We write the correction as

$$\Delta B\,(T(\tau_\mathrm{P})) = B\,(T_2(\tau_\mathrm{P})) - B\,(T_1(\tau_\mathrm{P}))$$

where our first-guess solution is $B\,(T_1(\tau_\mathrm{P}))$ and the updated estimate, after adding this correction, is $B\,(T_2(\tau_\mathrm{P}))$.

Writing eqtn. (9.6) for $B(T_1)$ and for $B(T_2)$ and subtracting gives

$$\Delta B\,(T(\tau_\mathrm{P})) = \frac{k_\mathrm{J}}{k_\mathrm{P}}\left[3\int_0^{\tau_\mathrm{P}} \frac{k_\mathrm{H}}{k_\mathrm{P}}\Delta H(t)\,\mathrm{d}t + 2\Delta H(0)\right] - \frac{\mathrm{d}(\Delta H(\tau_\mathrm{P}))}{\mathrm{d}\tau_\mathrm{P}}$$

where the 'ΔH' terms are corrections $H_2 - H_1$, and we have assumed that the ratios k_J/k_P, k_H/k_P are the *same* for the new estimate as for the first one.[7]

We know what the target value[8] is for the Eddington flux – it's just $\sigma T_{\rm eff}^4/(4\pi)$, which is therefore always our guess at H_2. Similarly, the target gradient, $d(H_2(\tau_P))/d\tau_P$ is zero, so

$$\begin{aligned}\frac{d(\Delta H(\tau_P))}{d\tau_P} &= \left(\frac{d(H_2(\tau_P))}{d\tau_P}\right) - \frac{d(H_1(\tau_P))}{d\tau_P} \\ &= B\left(T_1(\tau_P)\right) - \frac{k_J}{k_P} J_1(\tau_P). \end{aligned} \tag{9.3}$$

Our required temperature correction is therefore

$$\Delta B\left(T(\tau_P)\right) = \frac{k_J}{k_P}\left[3\int_0^{\tau_P} \frac{k_H}{k_P}\Delta H(t)\, dt + 2\Delta H(0)\right] - B\left(T_1(\tau_P)\right) + \frac{k_J}{k_P} J_1(\tau_P) \tag{9.7}$$

Eqtn. (9.7) allows us to compute the desired correction to the Planck function (or temperature structure), which can then be applied iteratively as follows:

0. Obtain a first estimate of the temperature structure $T(\tau)$ (e.g., from a grey-atmosphere solution);

1. From $T(\tau)$, compute $B_\nu\left(T(\tau)\right)$, and thence k_P (ab initio, or, more realistically, from a precomputed set of opacities $k_\nu(\rho, T)$).

2. With the source function in hand, compute the mean intensities $J(\tau_P)$ from the formal solution (eqtn. 7.23; cf. item 4 in §9.1.1, above), and thence k_J.

3. Similarly, $H_1(\tau_P)$ (hence k_H) and $H_1(0)$ can be computed (eqtns. 7.24, 7.25). We know the true frequency-integrated flux – it is $H = F/4\pi = \sigma T_{\rm eff}^4/4\pi$ (at all depths, in radiative equilibrium) – so we can write the required corrections to the current estimates, $H_1(\tau_P)$, as

   $$\Delta H(\tau_P) = H - H_1(\tau_P).$$

   at each depth τ_P (including $\tau_P = 0$).

4. Substituting into eqtn. (9.7) gives depth-dependent corrections to $B\left(T_1(\tau_P)\right)$, hence updated values, $B\left(T_2(\tau_P)\right)$ – which translates into an updated temperature structure, $T_2(\tau_P)$. Return to step 1 until convergence is achieved.

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[7]The opacities are updated *after* we have a new estimate of temperature structure, so we're always using values from the previous iteration. Nevertheless, we can safely assume that this will provide a pretty good estimate of the opacity *ratios,* which we can expect to be less sensitive to temperature than are the separate values.

[8]The iterative process won't converge on it immediately, because we don't have self-consistent opacities, so it remains a 'target' through the process.

Points to note:

- At depth, $J \rightarrow B$ (and $k_\mathrm{J} \rightarrow k_\mathrm{P}$). Hence the $\mathrm{d}(\Delta H)/\mathrm{d}\tau_\mathrm{P}$ term, eqtn. (9.3), tends to zero – just as for Λ iteration.
- However, the first [bracketed] term in eqtn. (9.7) gives rapid convergence at large optical depth, because the target H is known exactly.

Because eqtn. (9.7) uses an exact calculation of the the flux error to determine the correction, convergence is rapid (although it isn't achieved in a single iteration because the calculation is exact for only approximate estimates of physical parameters). This also ensures that the solution converges to the correction solution (i.e., correct F_0).