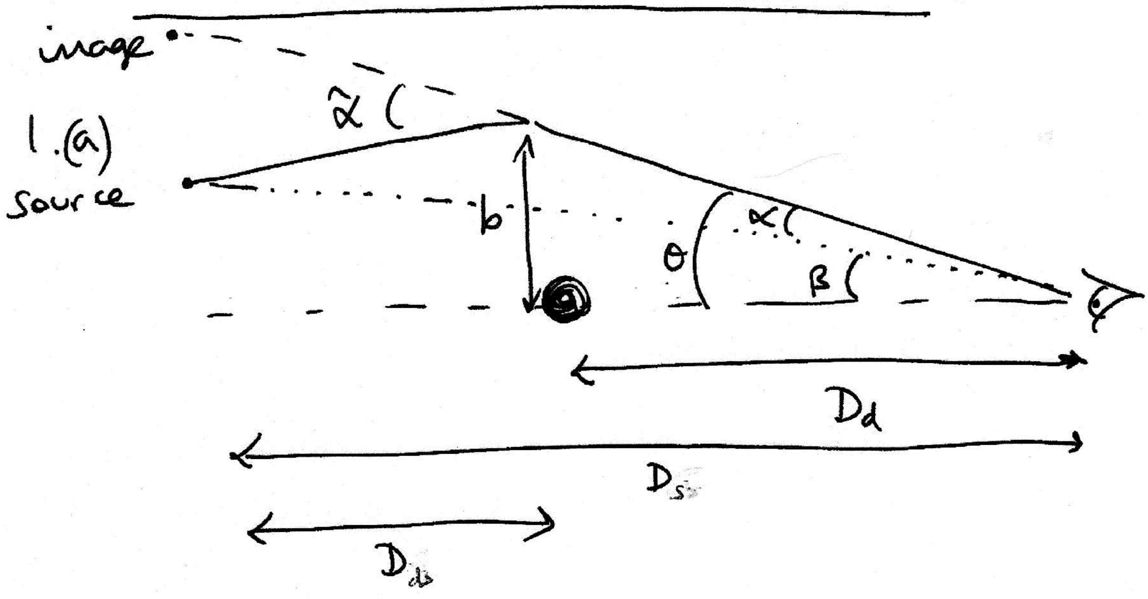
UCL PHAS 3136 [redacted] Problem sheet ~~3~~ 4 Solutions

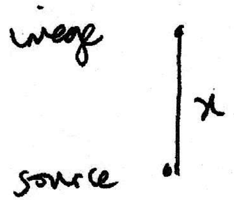
1.(a)



[2]

from GR: $\hat{\alpha} = \dfrac{4GM}{c^2 b}$ [1]

(b)



[they should draw this on the above diagram] [1]

$$x = \hat{\alpha} D_{ds}$$

$$x = \alpha D_s$$

$$\Rightarrow \quad \alpha = \frac{D_{ds}}{D_s} \hat{\alpha}$$ [2]

(c) $\cancel{\theta +}$ $\theta = \beta + \alpha$ [1]

$$\Rightarrow \theta = \beta + \frac{D}{D} \hat{\alpha}$$

$$= \beta + \frac{D}{D} \frac{4GM}{c^2 \underbrace{D_d \theta}_{b}}$$ [1]

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1 (c) contd.

multiply through by θ :

$$\theta^2 = \beta\theta + \frac{D_{ds}}{D_s}\frac{4GM}{c^2 D_d}$$

write as quadratic and complete the square:

$$\theta^2 - \beta\theta - \frac{4GM}{c^2}\frac{D_{ds}}{D_s D_d} = 0$$

$$\rightarrow \quad \theta = \frac{1}{2}\left[\beta \pm \sqrt{\beta^2 + 4\times\frac{4GM}{c^2}\frac{D_{ds}}{D_s D_d}}\right] \qquad [3]$$

(d) call θ_1, θ_2 solutions:

$$\left.\begin{aligned}\theta_1 &\equiv \frac{1}{2}\left[\beta + \sqrt{\beta^2 + \frac{16GM}{c^2}\frac{D_{ds}}{D_d D_s}}\right] \\ \theta_2 &\equiv \frac{1}{2}\left[\beta - \sqrt{\beta^2 + \frac{16GM}{c^2}\frac{D_{ds}}{D_d D_s}}\right]\end{aligned}\right\} \; [1]$$

~~rearrange θ_1 eqn to get close to mess.~~

$\cancel{((2\theta_1 - \beta)^2}$

need to eliminate β. Add eqns to find β in terms of θ_1, θ_2:

$$\theta_1 + \theta_2 = \frac{1}{2}\left[\beta + \sqrt{\dots}\right] + \frac{1}{2}\left[\beta - \sqrt{\dots}\right]$$

$$= \beta \qquad [1]$$

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Simplify eqns by subtracting one from the other:

$$\theta_1 - \theta_2 = \frac{2}{2}\sqrt{\beta^2 + \frac{16GM}{c^2}\frac{D_{ds}}{D_s D_d}}$$

$$\Rightarrow \quad (\theta_1 - \theta_2)^2 = \beta^2 + \frac{16GM}{c^2}\frac{D_{ds}}{D_s D_d}$$

subs $\beta = \theta_1 + \theta_2$

$$\Rightarrow \quad (\theta_1 - \theta_2)^2 = (\theta_1 + \theta_2)^2 + \frac{16GM}{c^2}\frac{D_{ds}}{D_s D_d}$$

rearrange

$$M = \frac{c^2}{16G}\frac{D_s D_d}{D_{ds}}\left((\theta_1 - \theta_2)^2 - (\theta_1 + \theta_2)^2 \right) \quad [3]$$

or marks for a good attempt or other versions of the above.

(N.B $\theta_2 < 0$)

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2. (a) size $\sim c \times \text{time}$ [1]

$\sim 3 \times 10^5$ km [1]

(b) size $> R_s \approx 2GM$ [1]

$= \dfrac{2GM}{c^2}$ [1]

$$M < \frac{\text{size} \times c^2}{2G}$$ [1]

$$= \frac{(3\times10^8)^2 \times 3\times10^8}{2\times 6\times10^{-11}}$$

$$= 5\times10^{34}\ \text{kg}$$

$$= \sim 2\times10^4\ M_\odot$$ [1]

(c) radiation pressure arguments:

gravitational attraction > radiation pressure [1]

$$\frac{GMm_p}{r^2} > \frac{L}{4\pi r^2}\frac{\sigma_T}{c}$$ [1]

$$\rightarrow M > \frac{L\sigma_T}{4\pi c G m_p}$$

$$> 0.1\ M_\odot$$ [1]

Yes $0.1\ M_\odot < M < 2\times10^4\ M_\odot$ (2)

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2 contd.

(d) flux density at radius r. $\sim \frac{L}{4\pi r^2}$

energy recieved by dust grain of radius a $\simeq \frac{L}{4\pi r^2} \pi a^2$ (1)

energy radiated $= \sigma T^4 \, 4\pi a^2$ [1]

in eqm received = radiated [1]

$\Rightarrow \quad \frac{L}{4\pi r^2} \pi a^2 = \sigma T^4 \, 4\pi a^2$ (1)

$$\Rightarrow \quad r = \sqrt{\frac{L}{\sigma T^4 \, 4 \times 4\pi}}$$

$$\simeq 5 \times 10^{-5} \text{ pc}$$ (1)

(3) • redshift $z = \frac{1}{a} - 1$

• age for EdS $H^2 = H_0^2 a^{-3}$

$$\therefore \left(\frac{\dot{a}}{a}\right)^2 = \frac{H_0^2}{a^3}$$

$$\therefore \frac{da}{dt} = \frac{H_0}{a^{1/2}} \quad \therefore a^{1/2} da = H_0 dt$$

$$\therefore \frac{2}{3} a^{3/2} = H_0 t + \cancel{c}^{\,0}$$

$$\therefore a = \left(\frac{3}{2} H_0 t\right)^{2/3}$$

• $T \propto \frac{1}{a}$ and $T_0 = 2.7K$ at $a = 1$ (CMB)

hence $\frac{T}{T_0} = \frac{a_0}{a} \quad \therefore \quad T = \frac{2.7K}{a}$.

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so if you know 1 of the above, translate.

a) matter - DE equality $H^2(a) = H_0^2 \left(\Omega_m a^{-3} + \Omega_\Lambda \right)$

$$\Omega_m a^{-3} = \Omega_\Lambda$$

$$a = \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{1/3}$$ [4] + [2] for filling in.

b) $T_{rec} = 3000\,K$ [2] for filling in

c) matter - rad eq. $z = 3500$ [2] for filling in

d) nucleosynthesis $E = 1\,MeV$. $T = \frac{E}{k_B} = \frac{1\,MeV}{8.6 \times 10^{-5}\,eV\,K^{-1}}$

$$= 1.1 \times 10^{10}\,K.$$

[1]

point d), cannot use $a = \left(\frac{3}{2} H_0 t\right)^{2/3}$ as EdS.

$$H^2 = H_0^2 a^{-4} \quad \therefore \quad \left(\frac{\dot a}{a}\right)^2 = H_0^2 a^{-4} \quad \therefore \quad \frac{\dot a}{a} = \frac{H_0}{a^2}$$

$$\therefore \quad \frac{da}{dt} = H_0 a^{-1}$$

$$\therefore \quad a\,da = H_0\,dt \quad \therefore \quad \frac{1}{2} a^2 = H_0 t$$

$$\therefore \quad a \propto t^{1/2}$$

$$\therefore \quad \frac{a}{a_0} = \left(\frac{t}{t_0}\right)^{1/2}$$

a_0 & t_0 from matter-rad eq. and $a \leftrightarrow t$ for nucleosynthesis.

[2]

[1] for filling the rest in

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4

$$\epsilon_R = \alpha T^4 \quad , \quad \alpha = \frac{\pi^2 k_B^4}{15 \hbar^3 c^3}$$

(a) $\rho_R = \epsilon_R / c^2$ [1]

(b)
$$\rho_{R0} = \frac{\pi^2 k_B^4}{15 \hbar^3 c^3} (2.7\,K)^4 / c^2$$

$$= \frac{\pi^2 \times (1.38 \times 10^{-23})^4 \; 2.7^4}{15 \times (1.06 \times 10^{-34})^3 \times (3 \times 10^8)^5} \; kg\, m^{-3}$$

$$= 4.38 \times 10^{-31} \; kg\, m^{-3}$$ [2]

To within 1 order of magnitude. Working + units must be shown to obtain full 2 marks.

(c)
$$\rho_{c0} = \frac{3 H_0^2}{8 \pi G}$$

$$= \frac{3 \times (100\, h \; km\, s^{-1}\, Mpc^{-1})^2}{8 \times \pi \times 6.67 \times 10^{-11} \; m^3\, kg^{-1}\, s^{-2}}$$

$$= 1.79 \times 10^{13} \; h^2 \, (km/Mpc)^2 \; kg\, m^{-3}$$

$$= 1.79 \times 10^{13} \; (10^3 \times 3 \times 10^{16} / 10^{-6})^{-2} \; h^2\, kg\, m^{-3}$$

$$= 2.0 \times 10^{-26} \; h^2\, kg\, m^{-3}$$

To within order of magnitude [1]
working [1]
units [1]

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~~X~~ 4 cont'd.

(d) matter - radiation equality occurs when

$$\rho_R = \rho_m$$

$$\Rightarrow \quad \rho_{R0}\, a^{-4} = \rho_{m0}\, a^{-3}$$

$a = \frac{1}{(1+z)}$ $\quad \Rightarrow \quad 1+z = \frac{\rho_{m0}}{\rho_{R0}}$

$\frac{\rho_{m0}}{\rho_{c0}} \equiv \Omega_m$ $\quad = \Omega_m \frac{\rho_{c0}}{\rho_{R0}}$ [3]

$$= \frac{\Omega_m \times 2.0 \times 10^{-26}\ h^2\,\mathrm{kg\,m^{-3}}}{4.38 \times 10^{-31}\ \mathrm{kg\,m^{-3}}}$$

$= 45\,000\ \Omega_m h^2$ to order of magnitude [2]

req'd for full 2 marks

(e) photon to baryon ratio $= \frac{\rho_{R0}}{3kT/c^2} \times \left(\frac{\rho_{B0}}{m_p}\right)^{-1}$

$\frac{\rho_{B0}}{\rho_{c0}} \equiv \Omega_b$ $\quad = \frac{\rho_{R0}}{\Omega_b\,\rho_{c0}} \times \frac{m_p}{3kT/c^2}$ [2]

$$= \frac{1}{\Omega_b h^2\ 45000} \times \frac{1.67 \times 10^{-27}\ \mathrm{kg}}{3 \times 1.38 \times 10^{-23}\ \mathrm{m^2\,kg\,s^{-2}\,K^{-1}} \times 2.7\,\mathrm{K} \times (3 \times 10^8\ \mathrm{m\,s^{-1}})^{-2}}$$

$$\simeq \frac{3 \times 10^7}{\Omega_b h^2}$$

$$\sim \frac{3 \times 10^7}{0.04 \times 0.7^2}$$

$$= 1.5 \times 10^9$$

$$\sim 10^9$$

To order of magnitude [2]

(5) If the universe was filled with gray dust.

- SN a of equal absolute magnitudes would appear dimmer than in the same universe without gray dust [2].

- given that Ω_Λ makes D_L larger and hence makes SNa appear dimmer [1]

- a universe with gray dust would weaken evidence for DE w/ SNa [1].