PHAS 3136 Problem Sheet 1 2010 Marking Scheme

1. Population III stars are formed from primordial gas (ie. that produced in the Big Bang)

$N \equiv$ total number of nuclei produced in Big Bang (in some volume)

$$\frac{N_{Li}}{N} = 1.6 \times 10^{-10}$$

$$\frac{N_{He}}{N} = 0.075$$

$$N_H + N_{He} + N_{Li} = N$$

Metallicity def^n

$$Z \equiv \frac{\text{mass of elements heavier than He}}{\text{total mass}} \quad [1]$$

$$= \frac{N_{Li}\, m_{Li}}{N_H m_H + N_{He} m_{He} + N_{Li} m_{Li}} \quad [1]$$

$$N_H = N - N_{He} - N_{Li} \quad [1]$$

$$\left.\begin{aligned} m_{He} &= 4 m_H \\ m_{Li} &= 7 m_H \end{aligned}\right\} [1]$$

$$\Rightarrow \quad Z = \frac{(N_{Li}/N)(m_{Li}/m_H)}{(1 - N_{He}/N - N_{Li}/N) + (N_{He}/N)(m_{He}/m_H) + (N_{Li}/N)(m_{Li}/m_H)}$$

$$= \frac{1.6 \times 10^{-10} \times 7}{(1 - 1.6 \times 10^{-10} - 0.075) + 0.075 \times 4 + 1.6 \times 10^{-10} \times 7}$$

$$= 9 \times 10^{-10} \sim 10^{-9} \quad [1]$$

PHAS 3136 Problem sheet 1 2010 marking scheme (2)

2 (a) Outer regions, no dark matter, no stars.

→ is as if all mass is at center of galaxy (Birkoffs theorem)

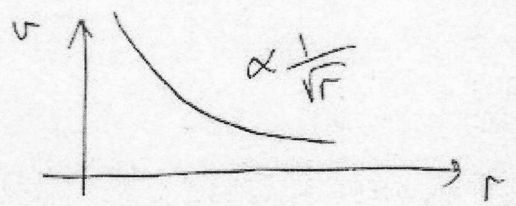
→ Circular velocity can be calculated from

centripetal force = gravitational force

$$m\frac{v^2}{r} = \frac{GMm}{r^2} \qquad [1]$$

where m is mass of test particle.

$$\Rightarrow v \propto \frac{1}{\sqrt{r}} \qquad [1]$$

→ 

PHAS 3136 Problem sheet 1 2010 marking scheme (3)

2(b)

$$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

$$M = \frac{rv^2}{G}$$

$$= \frac{(220 \times 10^3)^2}{7 \times 10^{-11}} \text{ kg m}^{-1}\ r$$

$$\simeq 5 \times 10^{20} \text{ kg m}^{-1}\ r \qquad [2]$$

$$\simeq \frac{5 \times 10^{20}}{2 \times 10^{30}} \times 3 \times 10^{19}\ M_\odot \text{ kpc}^{-1}\ r$$

$$\simeq 7 \times 10^{9}\ M_\odot \text{ kpc}^{-1}\ r$$

[either units are ok, but units ~~must~~ **must** be given for full marks.
Only 1 mark if e.g. $M = 5 \times 10^{20}\ r$.]

(c) we observe v = const wrt r

$\Rightarrow$ again using force balancing

$$\frac{mv^2}{r} = \frac{GMm}{r^\gamma} \qquad [1]$$

no dark matter $\Rightarrow$ M (mass enclosed) = const wrt r [1]
i.e. can assume all mass is in centre as above.

$\Rightarrow$ need $\gamma = 1$ for powers of r to match. [1]

PHAS 3136 Problem sheet 1 2010 marking scheme (4)

2 Cont'd

(d) $$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

where M = mass enclosed at radius r (within sphere)

$$M(r) = \int_0^r 4\pi r'^2 \rho(r')\, dr' \qquad [1]$$

assume $\rho(r) = \rho_0 r^\alpha$ and find ρ_0 and α

$$M(r) = \int_0^r 4\pi r'^2 \rho_0 r'^\alpha\, dr'$$

$$= \frac{4\pi \rho_0 r^{3+\alpha}}{3+\alpha}$$

cf result in (b) $\Rightarrow$ $3+\alpha = 1$ $\Rightarrow$ $\alpha = -2$ [1]

and $$\frac{4\pi\rho_0}{3-2} = 7\times10^9\ M_\odot\,\mathrm{Kpc}^{-1}$$

$$\rightarrow \quad \rho_0 = \frac{7\times10^9}{4\pi}\ M_\odot\,\mathrm{kpc}^{-1} \sim 5\times10^8\ M_\odot\,\mathrm{kpc}^{-1} \qquad [1]$$

$$\Rightarrow \quad \rho(r) = 5\times10^8\ M_\odot\,\mathrm{Kpc}^{-1}\ r^{-2} \qquad [1]$$

[OR $$\rho_0 = \frac{5\times10^{20}}{4\pi}\ \mathrm{kg\,m^{-1}}\ r^{-2} \sim 5\times10^{19}\ \mathrm{kg\,m^{-1}}\ r^{-2}$$

PHAS3136 Problemsheet 1 2010 Marking scheme (5)

2 contd

(e) $\rho(r) =$ const wrt r in center of galaxy. $\equiv \rho_0$

$$M(r) = \int_0^r 4\pi r'^2 \rho(r')\, dr'$$

$$= \frac{4\pi r^3}{3} \rho_0 \qquad \underline{\underline{\text{OR}}} \quad \propto r^3$$ [1]

Balance forces:

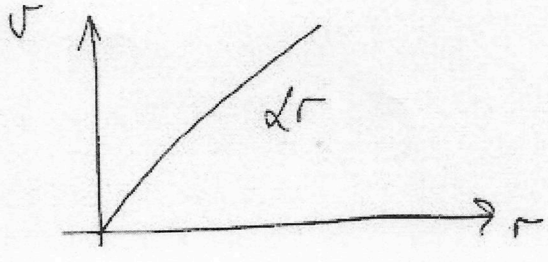
$$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

$$\Rightarrow \quad v^2 = \frac{G}{r} \frac{4\pi r^3}{3} \rho_0$$

$$= \frac{4\pi G}{3} r^2 \rho_0$$

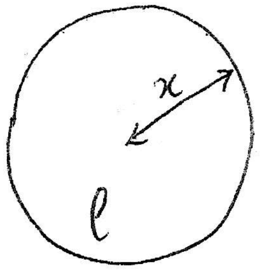
$$v = \sqrt{\frac{4\pi G \rho_0}{3}}\; r$$ [2]

[1 mark instead of 2 if just $v \propto r$]



PHAS3136 Problem Sheet 1 2010 (6)

1.



(a) $\left(E = \frac{4}{3}\pi x^3 \rho c^2\right)$

$$E = V\rho c^2$$ [1]

$$dE = \rho c^2 dV + Vc^2 d\rho$$ [1]

(b) $dS = 0 \Rightarrow \qquad dE = -p\,dV$

$$V = \frac{4}{3}\pi x^3$$

$$dV = 4\pi x^2 dx$$

$$x = ar$$

$$dx = r\,da$$ (since sphere expands with universe)

$$\Rightarrow \quad dV = 4\pi a^2 r^3 da$$

substitute into $\quad -p\,dV = \rho c^2 dV + Vc^2 d\rho$

$$-p\,4\pi a^2 r^3 da = \rho c^2 4\pi a^2 r^3 da + \frac{4}{3}\pi a^3 r^3 c^2 d\rho$$

take all differentials wrt time (divide through by dt):

$$-p\cancel{4\pi} a^2 \cancel{r^3} \dot{a} = \rho c^2 \cancel{4\pi} a^2 \cancel{r^3} \dot{a} + \frac{4}{3}\pi a^3 \cancel{r^3} c^2 \dot{\rho}$$

$\div \frac{a^3}{3}$: $$-3p\frac{\dot{a}}{a} = 3\rho c^2 \frac{\dot{a}}{a} + c^2\dot{\rho}$$

set $C = 1$ $$\rightarrow \quad \dot{\rho} + 3\frac{\dot{a}}{a}(p + \rho) = 0$$ [5]

(c) All quantities will now refer to radiation:

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{\rho}{3}\right) = 0$$

$$(\uparrow \text{subs } p = \tfrac{\rho}{3})$$

$$\rightarrow \frac{\dot{\rho}}{\rho} = -4\frac{\dot{a}}{a}$$

$$\rightarrow \int \frac{d\rho}{\rho} = -4\int\frac{da}{a}$$

$$\rightarrow \ln \rho = -4 \ln a + \text{const}$$

use $a=1$ when $\rho = \rho_0$ i.e. today

$$\rightarrow \ln \rho_0 = \text{const} \qquad (\text{since } \ln(1) = 0)$$

$$\rightarrow \ln\rho = -4\ln a + \ln \rho_0$$

$$\rightarrow \rho = \rho_0 a^{-4}$$ [4]

(d) $$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_R$$

$$= \frac{8\pi G}{3}\rho_{R0}a^{-4}$$

define $\Omega_R \equiv \frac{\rho_{R0}}{\rho_{c0}} = \frac{8\pi G}{3H_0^2}\rho_{R0}$

$$\rightarrow \left(\frac{\dot{a}}{a}\right)^2 = H_0^2\Omega_R a^{-4}$$

Today $\frac{\dot{a}}{a} = H_0$, $a=1$ $\rightarrow$ $H_0^2 = H_0^2\Omega_R \rightarrow \Omega_R = 1$

$$\rightarrow \left(\frac{\dot{a}}{a}\right)^2 = H_0^2 a^{-4}$$

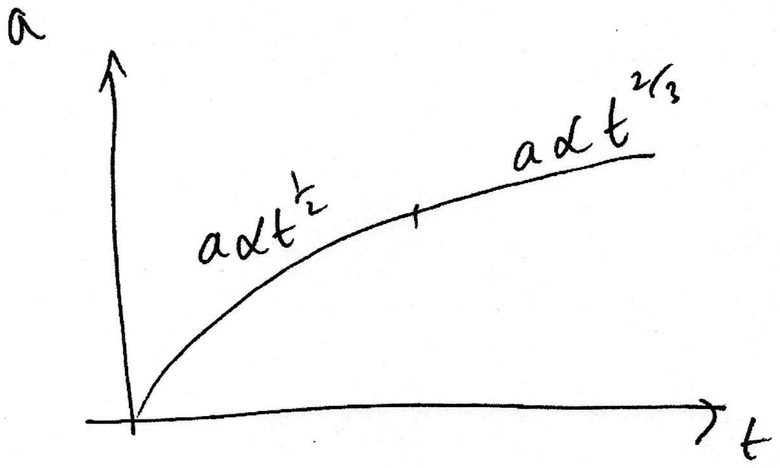
$$\rightarrow a^4\frac{\dot{a}^2}{a^2} = H_0^2 \quad \rightarrow \int a\,da = \int H_0\,dt$$

$$\rightarrow \frac{a^2}{2} = H_0 t + \text{const}$$

$a=0$ at $t=0$ $\rightarrow$ const $= 0$

$$\rightarrow a = (2H_0 t)^{\frac{1}{2}}$$ [5]

(e)



radiation dominated matter dominated [4]

2. $\rho = 0 \Rightarrow \Omega_m = 0$

$\cancel{\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho + H_0^2 \Omega_k a^{-2} + H_0^2 \Omega_\Lambda}$

use $1 = \Omega_m + \Omega_k + \Omega_\Lambda$

$\Rightarrow \quad \Omega_k = 1$ [1]

$\left(\frac{\dot a}{a}\right)^2 = H_0^2 a^{-2}$

$\Rightarrow \quad da = H_0 dt$

$a = H_0 t + \text{const}$

$a = 0$ at $t = 0 \rightarrow$ const $= 0$

$a = H_0 t$ [3]

age comes from $a = 1$ today $\Rightarrow t = \text{age} = \frac{1}{H_0}$

$= \frac{1}{70 \text{ km s}^{-1} \text{ Mpc}^{-1}}$

$= 15$ Gyr [1]