Observational cosmology: The Friedman equations 2

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UCL

OUTLINE
The Hubble parameter

After these lectures, you should be able to:

- Define the Hubble parameter H
- Sketch a(t) for k>0, k=0, k<0 assuming $\Lambda=0$
- Define ρ_c and comment on its value
- Define Ω_m, Ω_Λ, Ω_K
- Rearrange the Friedmann eqn in terms of H, Ω_m, Ω_Λ and Ω_K
- Show that $\Omega_K = 1 - \Omega_m - \Omega_\Lambda$
- Derive and discuss a(t), adot(t), addot(t) if $\Omega_m=1$, $\Omega_K=0$
- Derive and discuss a(t), adot(t), addot(t) for a radiation dominated universe
- Sketch a(t) for a mixture of matter and radiation
- Sketch a(t) for a matter dominated universe with k.ne.0 but $\Lambda=0$
- Derive a(t) for a Λ dominated universe with k=0
- Derive the age of an EdS Universe
- Discuss the effect of a non-zero cosmological constant on the age of the Universe
- Estimate the redshifts of matter-radiation, matter-DE equality

Hubble's law derived

- The flow caused by the expansion can be written as:
- So using again the fact that r = a(t) x and x is a co-moving.
- Note that H is a constant in space but it is not a constant in time.
- The Hubble constant is the value today. In other times H(z).

$$v = \frac{\mathrm{d}r}{\mathrm{d}t}$$

$$v = \frac{\dot{r}}{r} r$$

$$v = \frac{\dot{a}}{a} r = Hr$$

A new version of the Friedman equation

- If we re-write the Friedman equation with matter radiation and dark energy.
- This way we can write down a fractional contribution to each component in the Universe.
  - Omega=rho/rho_c
  - Where:
- Today:
  - OmegaL = 0.75
  - OmegaM = 0.25
  - Omega_r ~ 0.0
  - Omgea_k ~ 0.0

$$H^2 = \frac{8\pi G}{3}(\rho_m + \rho_r) - \frac{kc^2}{a^2} + \frac{\Lambda}{3}$$

$$H^2 = \frac{8\pi G}{3}(\frac{\rho_{m0}}{a^3} + \frac{\rho_{r0}}{a^4}) - \frac{kc^2}{a^2} + \frac{\Lambda}{3}$$

$$H^2 = H_0^2\left(\frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4} + \frac{\Omega_k}{a^2} + \Omega_\Lambda\right)$$

$$\rho_c = \frac{3H_0^2}{8\pi G}$$

$$\Omega_m + \Omega_r + \Omega_k + \Omega_\Lambda = 1$$

The Standard CDM model of the 80s

- If we take the Friedman equation

$$H^2 = H_0^2 \left(\frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4} + \frac{\Omega_k}{a^2} + \Omega_\Lambda \right)$$

- Assuming only matter in the Universe:
- We have the following Hubble parameter as a function of time

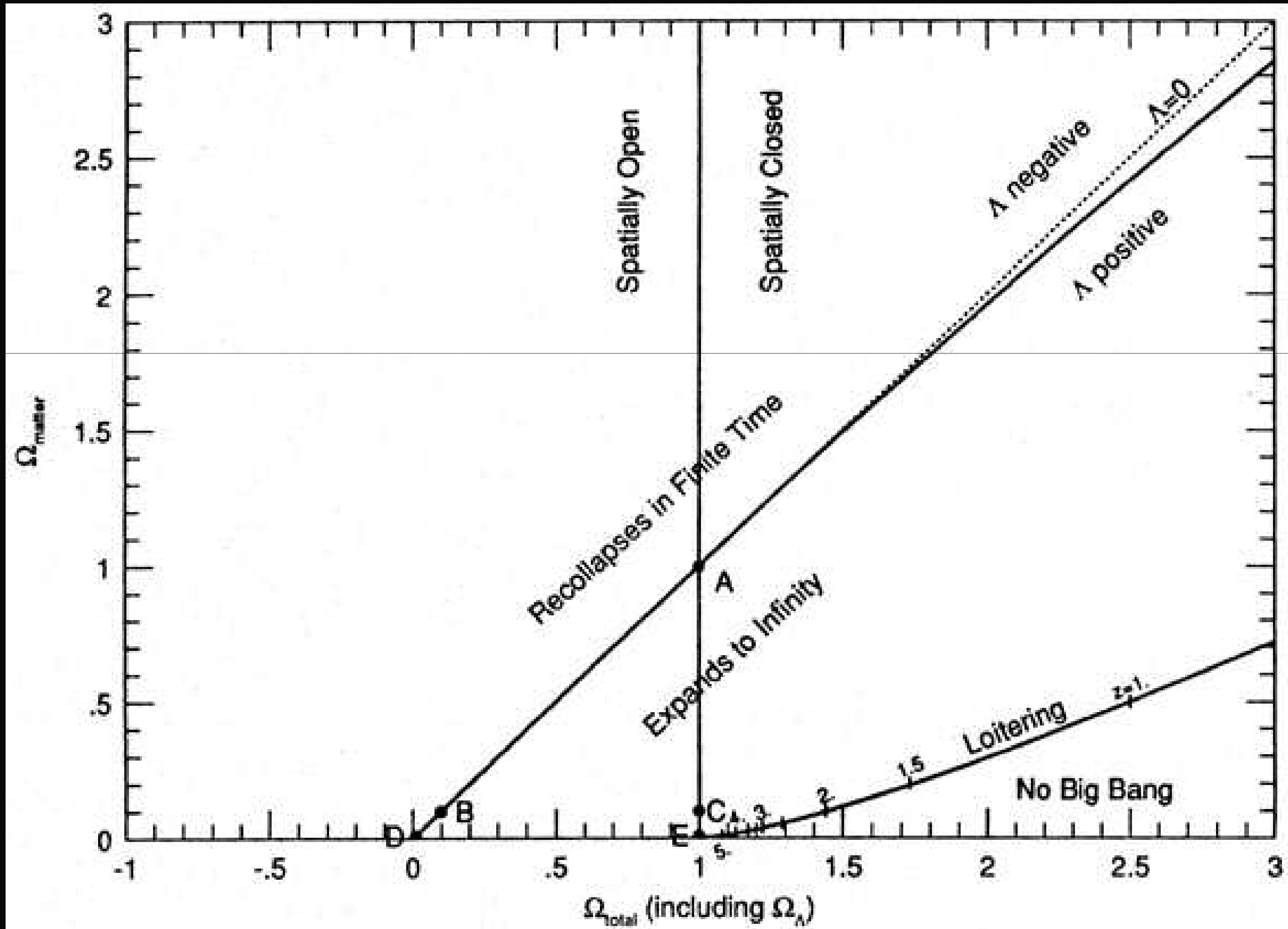
$$H^2 = H_0^2 \left(\frac{\Omega_m}{a^3} + \frac{1 - \Omega_m}{a^2} \right)$$

$$H^2 = H_0^2 (1+z)^2 \left(\Omega_m (1+z) + 1 - \Omega_m \right)$$

- The standard CDM has omega matter equal to one

$$H(z) = H_0 (1+z) \sqrt{1 + \Omega_m z}$$

Expansion of the universe:



Bouncing Universes:

- If dark energy is too dominant, the Universe has no big bang.
- We can show that in these cases the universe retracts from infinity then bounces back to re-expand.
- This happens if:

$$\Omega_\Lambda > 4\Omega_m \left\{ \mathrm{coss} \left[\frac{1}{3} \mathrm{coss}^{-1} \left(\frac{1-\Omega_m}{\Omega_m} \right) \right] \right\}^3$$

- And in these Universes there is a maximum redshift given by:

$$z > 2\mathrm{coss} \left[\frac{1}{3} \mathrm{coss}^{-1} \left(\frac{1-\Omega_m}{\Omega_m} \right) \right] - 1$$

- Where coss is cos if the matter density is bigger than 0.5 and cosh if it is smaller than 0.5

The evolution of the redsfhit.

- Is the redshift a constant in time?
  - How much does it change?
- Lets write down it's definition:
- Substituting the original definition of the redsfhit we have:
- This leads to the following general expression:
- In the case of a Universe with matter only we have:
- This leads to the redsfhit evolution being around one part in 10^8.

$$1+z=\frac{a(t_o)}{a(t_e)}=\frac{\mathrm{d}t_o}{\mathrm{d}t_e}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t_o}=\frac{\mathrm{d}a(t_o)}{\mathrm{d}t_o}\frac{1}{a(t_e)}-\frac{a(t_o)}{a^2(t_e)}\frac{\mathrm{d}a(t_e)}{\mathrm{d}t_e}\frac{\mathrm{d}t_e}{\mathrm{d}t_o}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t_o}=\frac{\dot{a}(t_o)}{a(t_o)}\frac{a(t_o)}{a(t_e)}-\frac{\dot{a}(t_o)}{a(t_e)}\frac{a(t_o)}{a(t_e)}\frac{1}{1+z}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t}=(1+z)H_0-H(z)$$

$$\frac{\mathrm{d}z}{\mathrm{d}t}=(1+z)H_0(1-\sqrt{1+\Omega_m z})$$

Age redshift relation

- If we write down the Friedman equation:
- The Friedman equation relates the scale factor to the time.
- We can rearrange the time part and write down an expression for the time as a function fo the scale factor.
- This gives the age of the Universe.

$$H^2 = H_0^2 \left(\frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4} + \frac{\Omega_k}{a^2} + \Omega_\Lambda \right)$$

$$H^2 = \left(\frac{\dot{a}}{a} \right) = \left(\frac{\mathrm{d}a}{\mathrm{d}t} \right)^2 \frac{1}{a^2}$$

$$T_0 = \int \mathrm{d}t = \int \frac{\mathrm{d}a}{aH(a)}$$

The big crunch?

- Lets calculate the age of the Universe if we have an Universe which is closed and has only matter:
- By writing down a scale factor derivative we can see that:
- The leads to a maximum scale factor if the lhs is equal to zero.
- So we have a maximum scale factor if matter density is bigger than one.
- Given that the expansion is reversible this means a big crunch at twice that scale factor.

$$H^2 = H_0^2 \left(\frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4} + \frac{\Omega_k}{a^2} + \Omega_\Lambda \right)$$

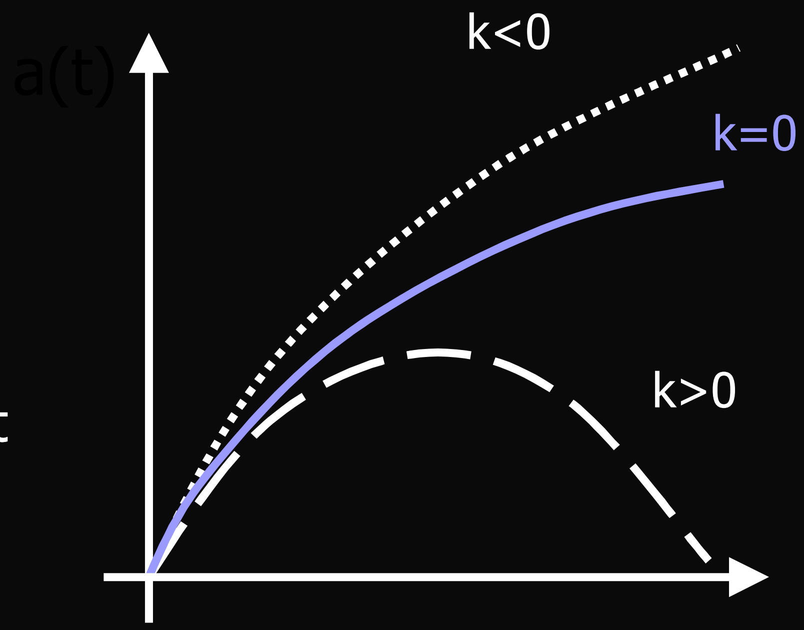
$$H^2 = \left(\frac{\dot{a}}{a} \right) = \left(\frac{\mathrm{d}a}{\mathrm{d}t} \right)^2 \frac{1}{a^2}$$

$$\frac{\mathrm{d}a}{\mathrm{d}t} = H_0 \sqrt{\frac{\Omega_m}{a} + 1 - \Omega_m}$$

$$a_{\mathrm{max}} = \frac{\Omega_m}{\Omega_m - 1}$$

Sketch a(t) for k>0, k=0, k<0 assuming $\Lambda=0$, $\rho_R=0$

- Late times:
  $-k/a^2$ term dominates
- k<0
  - $\dot{a}^2$ positive always
  - Universe expands forever
- k>0
  - adot becomes zero at some point
  - Universe stops expanding
  - adot imaginary → cyclic
- k=0
  - stops expanding at t=infinity
- Analogous to throwing a ball in the air, escape velocity.



Derive and discuss $a(t)$, $\dot{a}(t)$, $\ddot{a}(t)$ for an Einstein de-Sitter (EdS) universe

- EdS $\leftrightarrow$ $\Omega_k = 0$, $\Omega_\Lambda = 0$ ($\Omega_R = 0$)
- From $1 = \Omega_m + \Omega_k + \Omega_\Lambda \Rightarrow \Omega_m = 1$
- Friedman eq^n becomes: $\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 a^{-3}$
- Solve the Fr. eqn, to find $a(t)$:

$$\frac{\dot{a}}{a} = H_0 a^{-3/2}$$

Multiply through by $a^{3/2}$, $\dot{a} = \frac{da}{dt}$ $\rightarrow$ $a^{1/2}\, da = H_0\, dt$

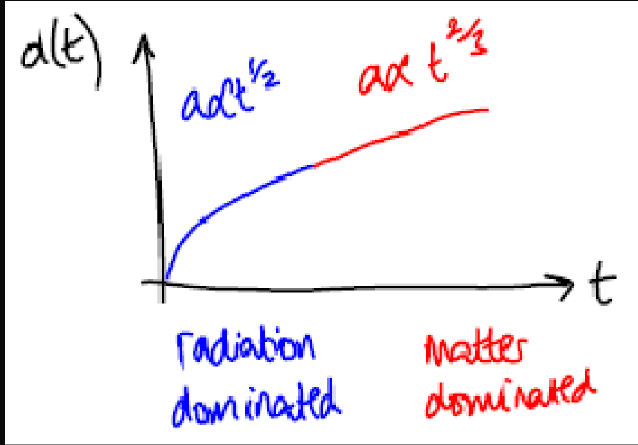
$$\Rightarrow \frac{2}{3} a^{3/2} = H_0 t + \text{const}$$

Assume Big Bang, always matter dominated (by def^n of EdS)
$\Rightarrow$ when $t=0$, $a=0$ $\Rightarrow$ const $=0$

Rearrange: $a = \left(\frac{3}{2} H_0 t\right)^{2/3}$ $\|$

Differentiate wrt t: $\dot{a} = \left(\frac{3}{2} H_0\right)^{2/3} \frac{2}{3} t^{-1/3}$ $\|$

" $\ddot{a} = -\frac{1}{3}\frac{2}{3}\left(\frac{3}{2} H_0\right)^{2/3} t^{-4/3}$ $\|\|$



Derive and discuss $a(t)$ for the Milne model

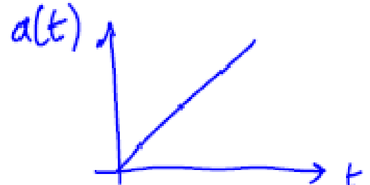
- Milne model $\leftrightarrow$ $\Omega_M = 0$, $\Omega_\Lambda = 0$ ($\Omega_R = 0$) $\leftrightarrow$ empty
- Friedmann eq^n: $\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \Omega_k a^{-2}$
- Solve for $a(t)$: $\frac{\dot{a}}{a} = H_0 \Omega_k^{1/2} a^{-1}$

Multiply by a: $\dot{a} = H_0 \Omega_k^{1/2}$ $\dot{a}$ constant...

$$da = H_0 \Omega_k^{1/2}\, dt$$

$$a = H_0 \Omega_k^{1/2} t + \cancel{\text{const}}$$ Big bang

$a \propto t$ expands freely forever

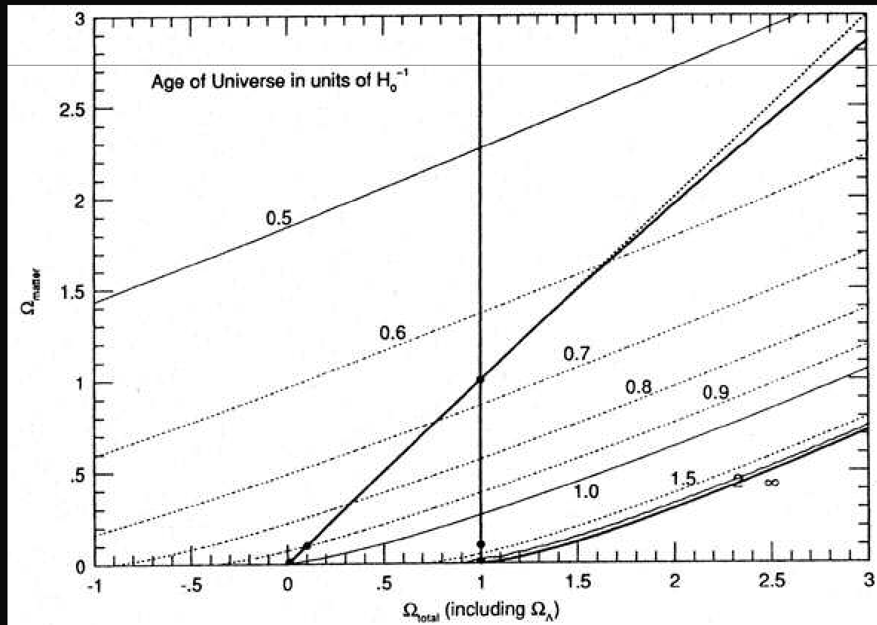


Age-redshift relation and Dark energy

- How does the age of the Universe change as a function of cosmological parameters:
- Higher matter density
  - Smaller age!!!
- Higher dark energy density
  - Larger age!!!
- First hint that a cosmological constant was needed.
  - Age of globular clusters
- http://nedwww.ipac.caltech.edu/level5/Carroll/frames.html

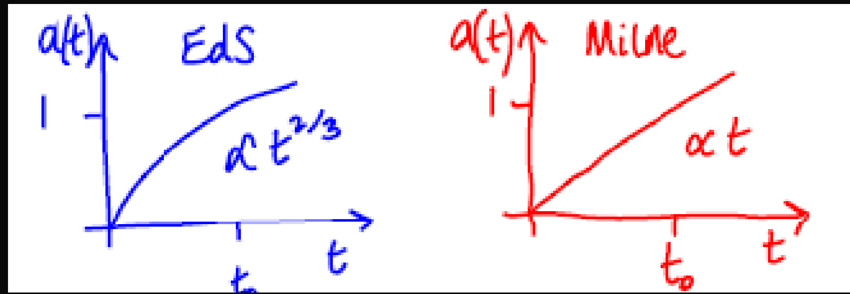
$$T_0 = \int \mathrm{d}t = \int \frac{\mathrm{d}a}{aH(a)}$$

$$H^2 = H_0^2 \left(\frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4} + \frac{\Omega_k}{a^2} + \Omega_\Lambda \right)$$

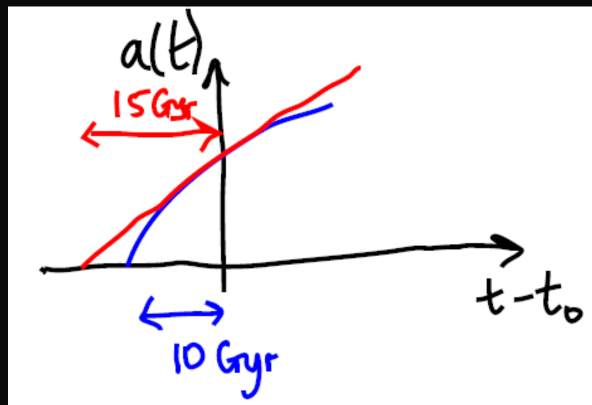


Explain why decreasing Ω_m increases the age of the Universe

- Consider two extremes:
  - EdS $\Omega_m = 1$
    a / $t^{2/3}$
  - Empty Universe $\Omega_m = 0$
    Ω_K=1 a / t



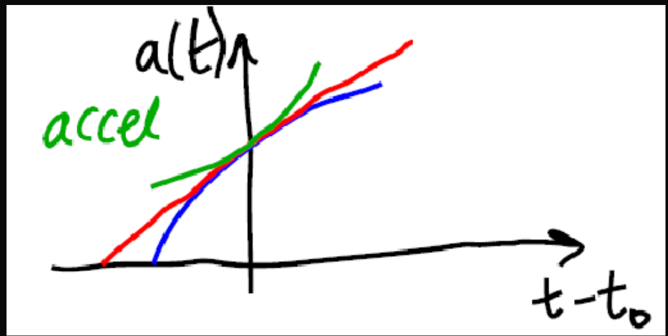
- Extra matter causes more deceleration, bending a(t) over
- Plot a(t) against t-t_0
- Constraints:
  - a(t_0) = 1
  - adot(t_0) = H_0 (ie. fixed)



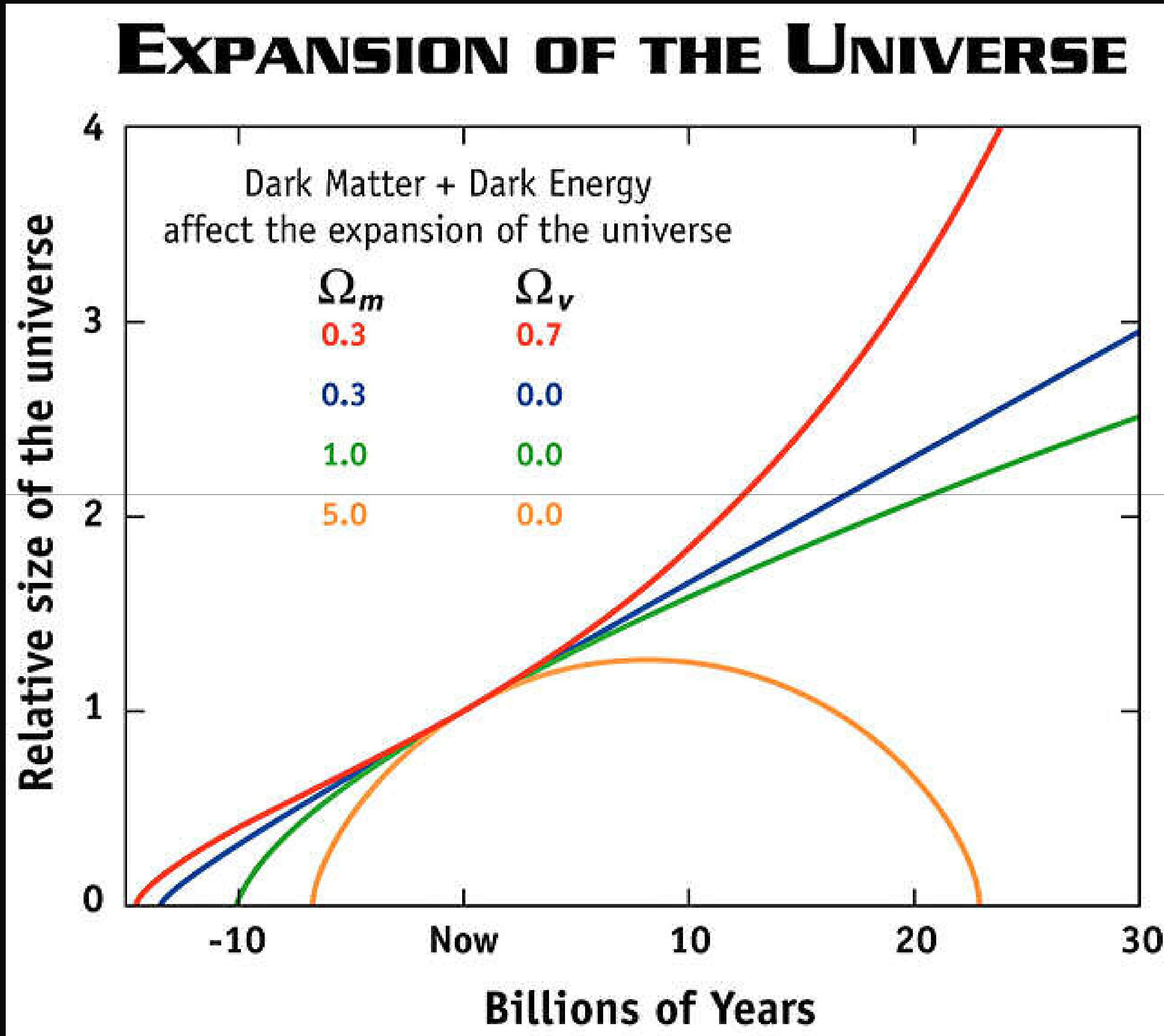
- t_0 is smaller for EdS
- Gradual transition as Ω_m is varied between models

Explain why increasing Ω_Λ increases the age of the Universe

- Ω_Λ causes acceleration at the present day
- Acceleration $\leftrightarrow$ a(t) curving upwards
- Consider effect on plot of a(t) versus $t\text{-}t_0$
  - Tendency to increase the age



- Consider an extreme case: Ω_Λ=1 model
  - Age of the Universe is infinite!
- Could now sketch age as a function of Ω_m



map.gsfc.nasa.gov/m_uni/101bb2_1.html

OUTLINE
Distances and Supernovae(Λ)

After these lectures, you should be able to:

- Define the integrated co-moving distance D and be able to calculate it for simple cosmologies
- Define the Luminosity Distance D_L
- Write down D_L in terms of D and justify qualitatively
- Explain what is meant by the term "standard candle"
- Discuss Type Ia supernovae (SNIa) as standard candles
- Discuss constraints on D_L and Ω_m and Ω_Λ from SNIa
- Comment on the future prospects

The Luminosity Distance D_L

- Effective distance D_L to an object
  - such that flux S received by us obeys
  - $S = L / (4 \pi D_L^2)$
  - L= luminosity of the object

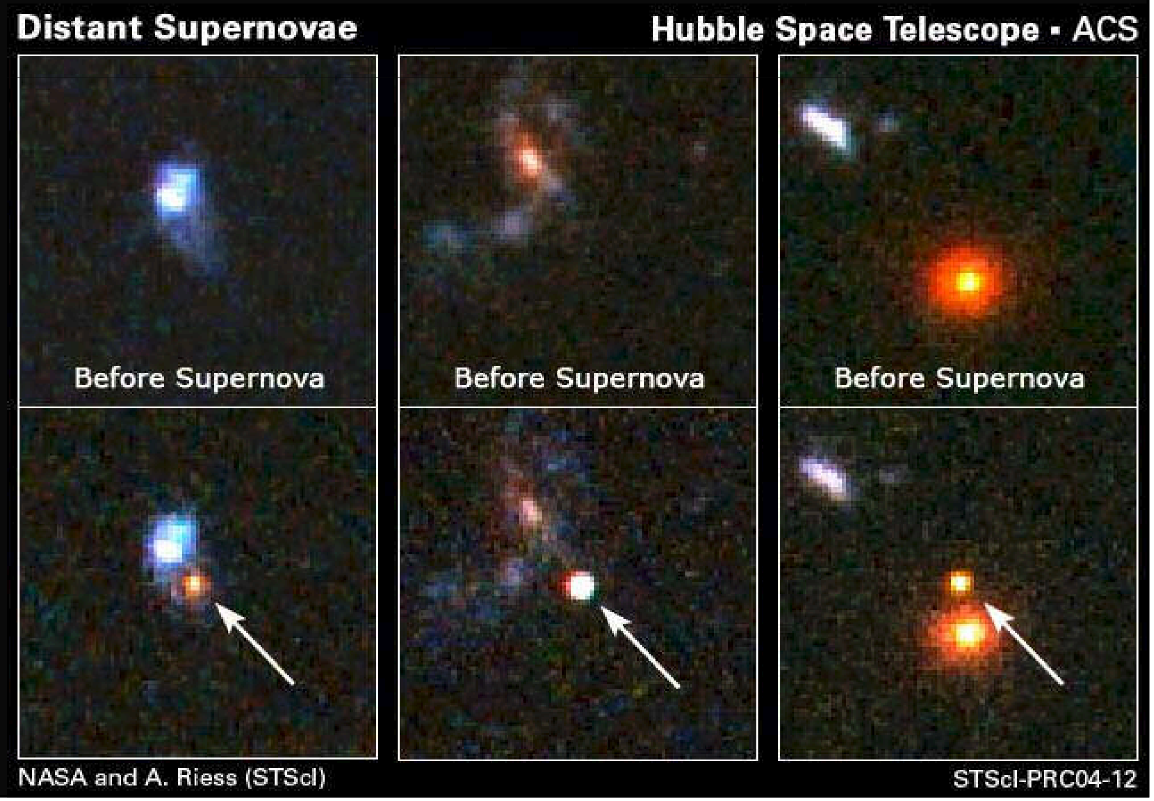
- Why wouldn't D_L = actual distance?
  - Curved geometry
  - Expanding universe
- Long derivation gives: $D_L = D (1+z)$

- Can think of this as a modification cf $D_L=D$, where S is reduced on two counts:
  - Photons lose energy / (1+z)
  - and arrive less frequently / (1+z)

Type Ia Supernovae

- Exploding star, briefly as bright as an entire galaxy
- Characterized by no Hydrogen, but with Silicon
- Gains mass from companion until undergoes thermonuclear runaway

Standard explosion of a WD reaching the Chandrasekhar limit



Fitting lightcurves: 2 methods

MLCS2k2

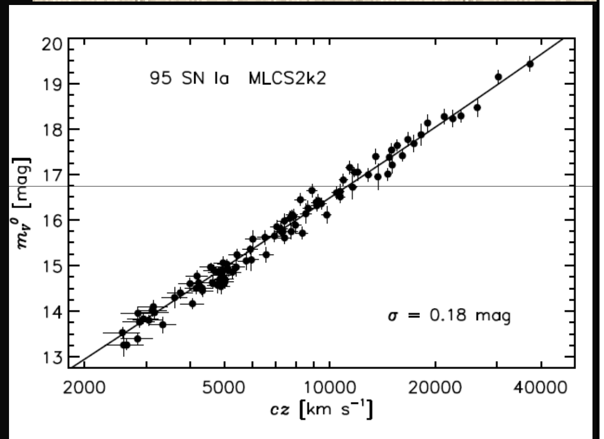
$$\vec{m}_X(t-t_0) = \vec{M}^0_X + \mu_0 + \boxed{\vec{\zeta}_X}\boxed{(\alpha_X + \beta_X/R_V)}A^0_V + \boxed{\vec{P}_X\Delta + \vec{Q}_X\Delta^2},$$

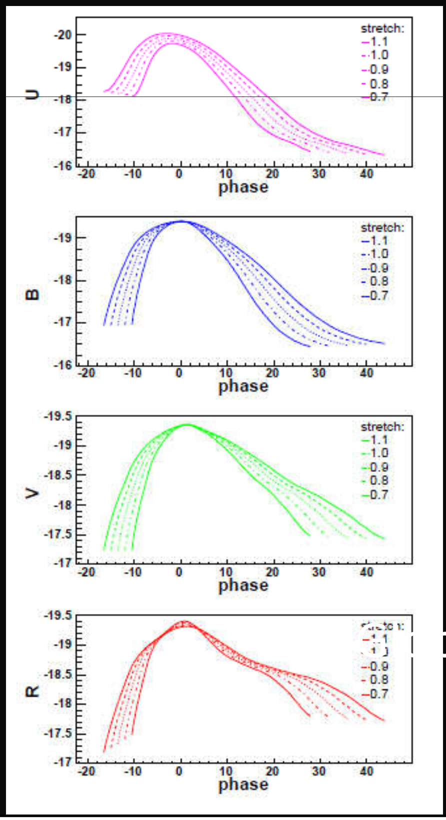
Extinction dependence

$$\chi^2 = \vec{r}^T C^{-1}\vec{r}, \qquad \vec{\zeta}_X \equiv \frac{\vec{A}_X}{A^0_X}, \qquad \frac{A^0_X}{A^0_V}$$

Second order correction in magnitude

Training set of 37 nearby SN





SALT

Aims to fit fluxes instead

Filter

$$f_{SN}(p,z,T) = \boxed{f_0}(1+z)\int \boxed{\phi}(p,\lambda,\boxed{s},\boxed{c})\,\frac{\lambda}{h\,c}\,\boxed{T}(\lambda(1+z))\,d\lambda$$

$$\mathbf{c} = (B-V)_{max} + 0.05$$

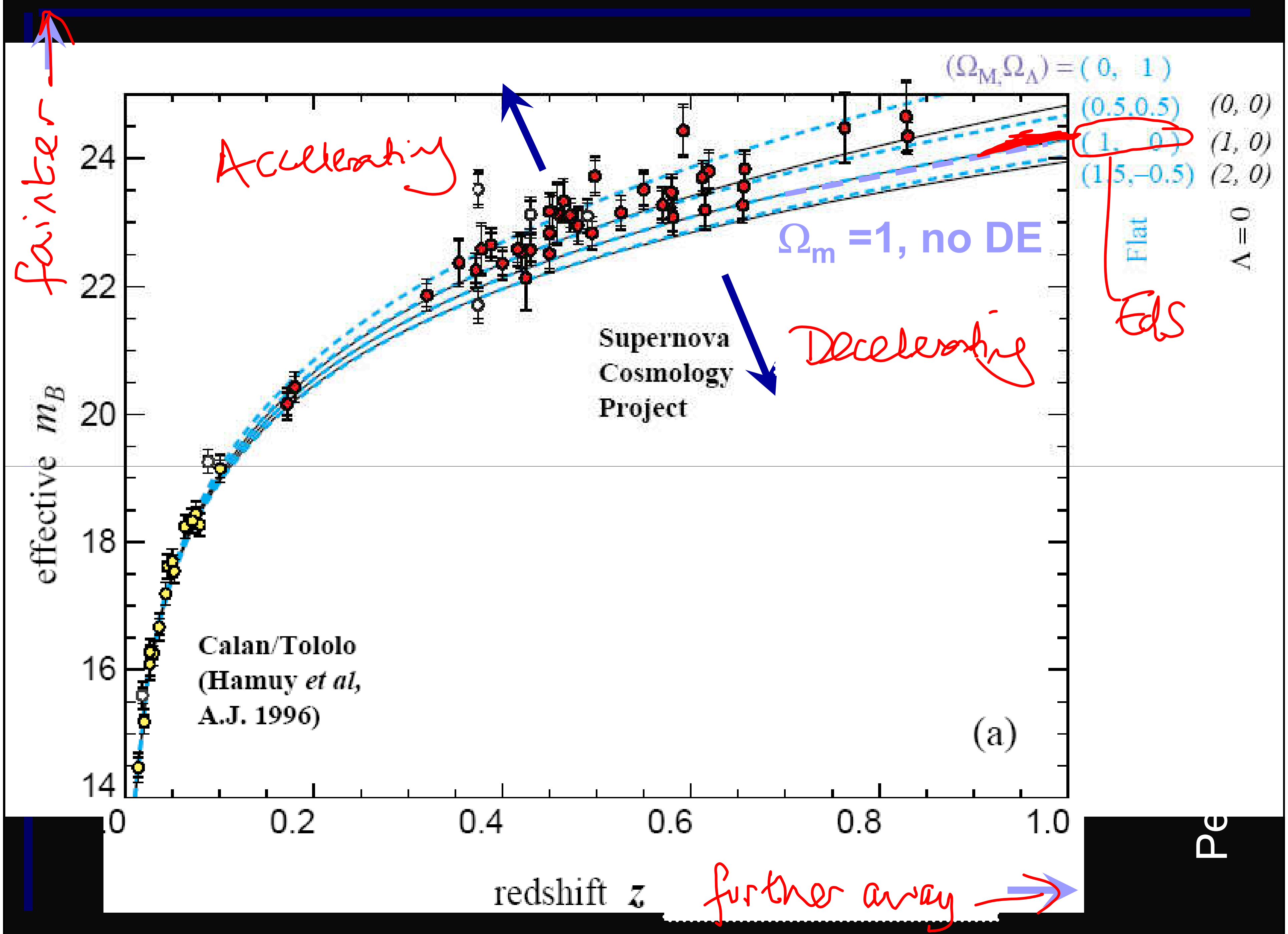
bal intensity parameter $d_L(z)^{-2}$
actual peak flux in B

Stretch factor

Model of SN energy per unit of wavelenght

Placing constraints on cosmology using data

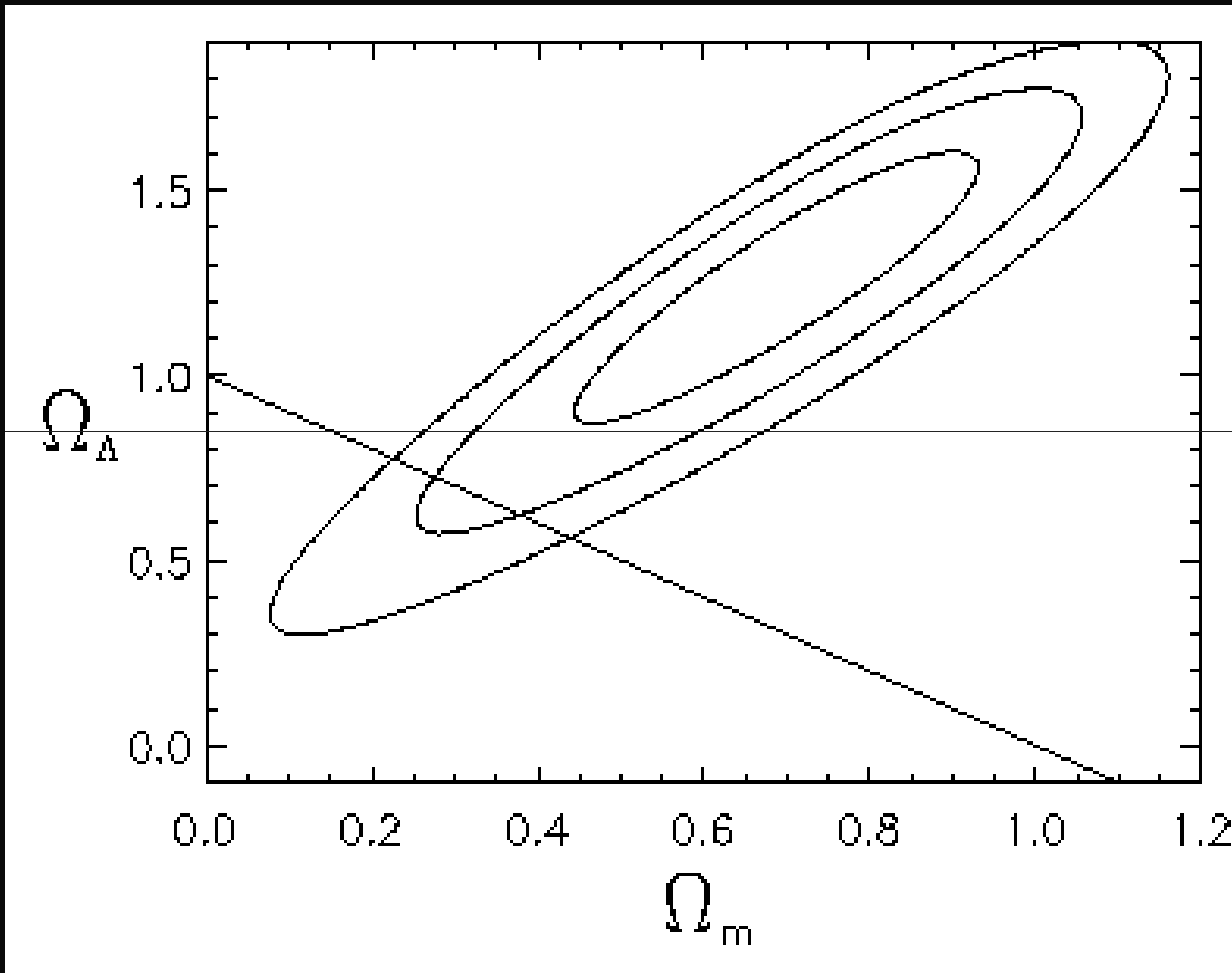
- Try all different values for cosmological parameters
  - e.g. Ω_Λ=0, 0.1, 0.2 ... 1 and Ω_m = 0,0.1,0.2.. 1
- Compare predictions with observations to get probability of trial parameter values
  - e.g. χ^2 statistic with Probability = $e^{-\chi^2/2}$
- Plot contours containing 68%, 95%, (99%)of probability



fainter
Accelerating
Decelerating
EdS
further away
$(\Omega_M, \Omega_\Lambda) = (0, 1)$
(0.5,0.5)
(1, 0)
(1.5,–0.5)
(0, 0)
(1, 0)
(2, 0)
Flat
$\Lambda = 0$
Ω_m =1, no DE
Supernova Cosmology Project
Calan/Tololo (Hamuy et al, A.J. 1996)
(a)
effective m_B
redshift z
14
16
18
20
22
24
0.2
0.4
0.6
0.8
1.0

Latest constraints from SN

Dark energy density



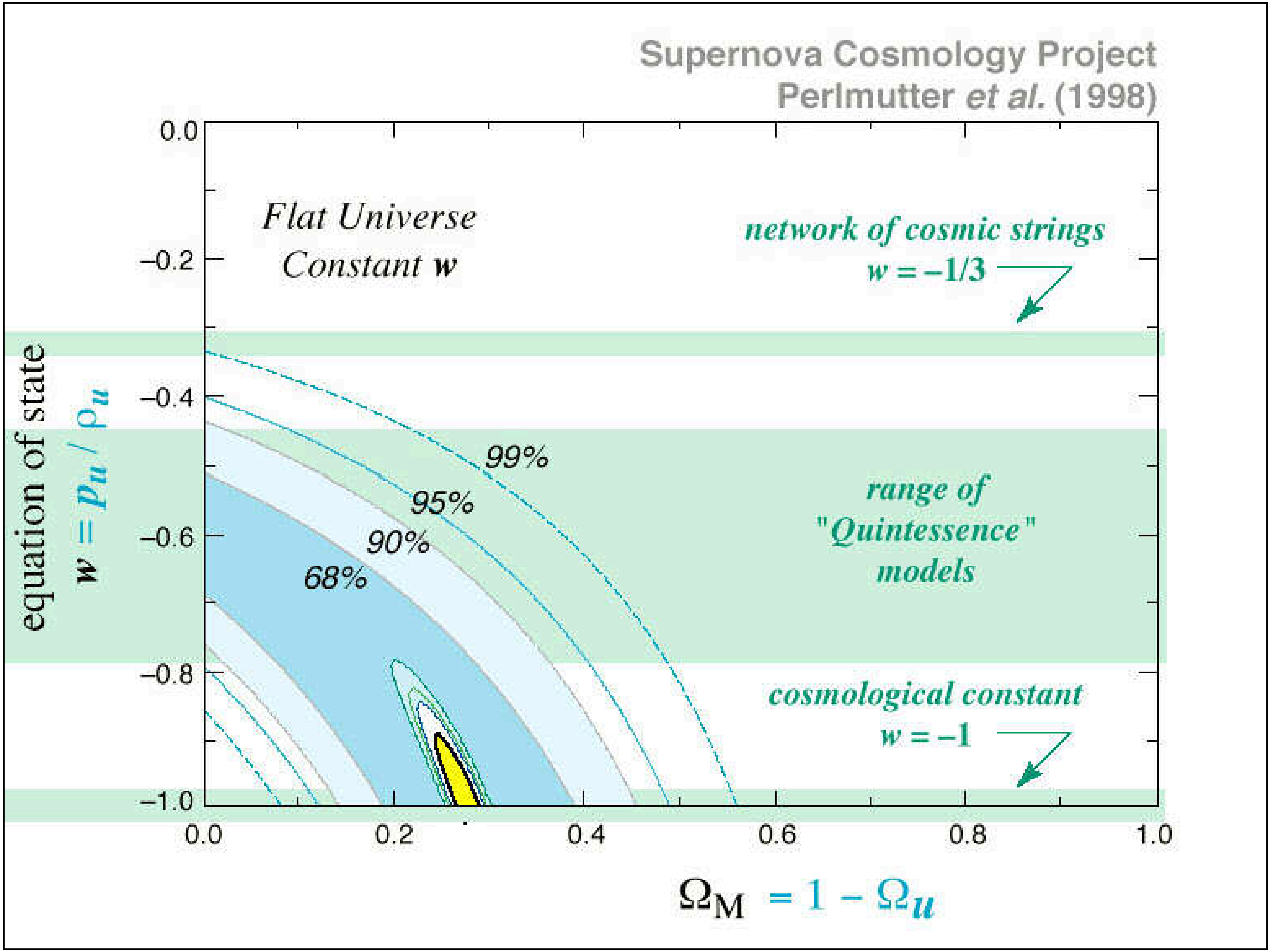
Dark matter density

Barris et al 2004

Future supernova data

- Ongoing
  - SNFactory, measure ~300 SNIa at z~0.05
  - CFHTLS
  - Carnegie supernova program

- Future
  - Pan-STARRS ~ 2011
  - Dark Energy Survey (DES) ~2011
  - SNAP/JDEM
    - Would measure ~2000 SNIa $0.1<z<1.7$
    - >2017
  - LSST
    - >2015


Supernova Cosmology Project
Perlmutter et al. (1998)
Flat Universe
Constant w
network of cosmic strings
w = −1/3
range of
"Quintessence"
models
cosmological constant
w = −1
99%
95%
90%
68%
equation of state
w = p_u / ρ_u
Ω_M = 1 − Ω_u
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
0.0
0.2
0.4
0.6
0.8
1.0

Integrated co-moving distance

- Crops up everywhere
  - Often written D

element of physical distance

$$D \equiv \int \frac{c\,dt}{a(t)}$$

element of comoving distance

$$\dot{a} = \frac{da}{dt} \Rightarrow dt = \frac{da}{\dot{a}}$$

$$D = c\int \frac{da}{\dot{a}\,a} = c\int \frac{a^{-2}\,da}{(\dot{a}/a)}$$

$$= c\int \frac{a^{-2}\,da}{\sqrt{\Omega_m a^{-3} + \Omega_k a^{-2} + \Omega_\Lambda}}$$

Calculate D from the Big Bang to the present day t_0 for Einstein de-Sitter.

EdS: $a(t) = \left(\frac{t}{t_0}\right)^{2/3}$

$$D = \int_{t=0}^{t=t_0} \frac{c\,dt}{a(t)}$$

$$= c\int_{t=0}^{t=t_0} t_0^{2/3}\, t^{-2/3}\, dt$$

$$= c\, t_0^{2/3} \left[\, 3\, t^{1/3} \right]_0^{t_0}$$

$$= 3ct_0$$

Why not just ct_0 ?

Universe expansion does part of the work for the light!

Distances: the angular diameter distance

- We define the angular diameter distance:
- This is constructed in such a way as to preserve the variation of angular size of an object with its distance.
- Let Dp be the proper distance of a source at r at time t.
- Then the angular diameter distance is equal to :
- Compared to Dl

$$D_a = \frac{D_p}{\Delta\theta}$$

$$D_p = ar\Delta\theta$$

$$D_a = ar$$

$$D_l = (1+z)^2 D_a$$

$$\begin{array}{ll} r = (1/\sqrt{k})\sin(\sqrt{k}\chi) & k > 0 \\ r = \chi & k = 0 \\ r = (1/\sqrt{|k|})\sinh(\sqrt{|k|}\chi) & k < 0 \end{array}$$

Distances:

- We define the *comoving distance D* by

$$D \equiv R_0 r(a) = c \int_{t(a)}^{t_0} \frac{dt'}{a(t')}$$

- We can define the *angular diameter distance* Da such that the transverse distance = D_a x dθ:

$$D_A = a R_0 S_k(r)$$

- And the *luminosity distance* D_l, such that F = L /(4πD_l^2)

$$D_L = a^{-1} R_0 S_k(r)$$

- Pre-factors of *a* come from the redshift of photons and their slower arrival rate.

Horizons: Particle and event

- We know that for light we have:
- So if we change to co-moving coordinates:
- If we calculate the co-moving coordinate travelled by a light-ray since the beginning of the Universe to time t.
- If we assume a matter dominated Universe we have the scale factor going as t to the power of 2/3.
- The event horizon encloses all the particles which in principle can be reached.

$$\frac{\mathrm{d}r}{\mathrm{d}t} = c$$

$$\mathrm{d}r = a(t)\mathrm{d}x$$

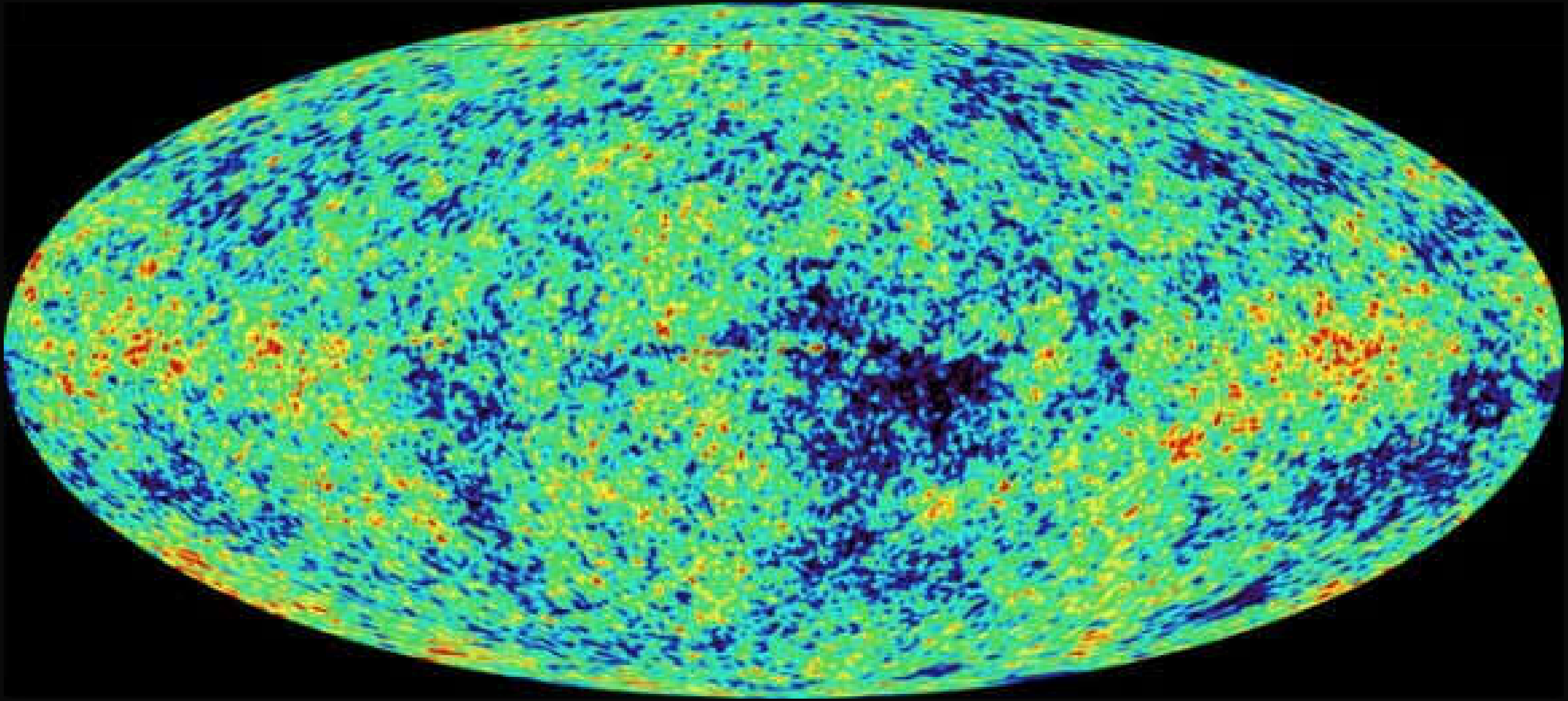
$$x_p = \int_0^{x_0} \mathrm{d}x = \int_0^{t_0} \frac{c\mathrm{d}t}{a(t)}$$

$$x_p = \int_0^{t_0} \frac{c\mathrm{d}t}{t^{2/3}} = 3ct$$

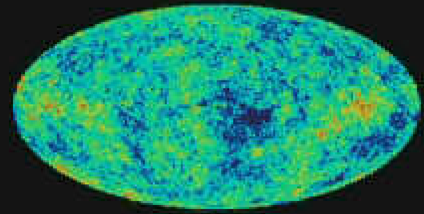
$$x_e = \int_{t_0}^{\infty} \frac{c\mathrm{d}t}{a(t)}$$

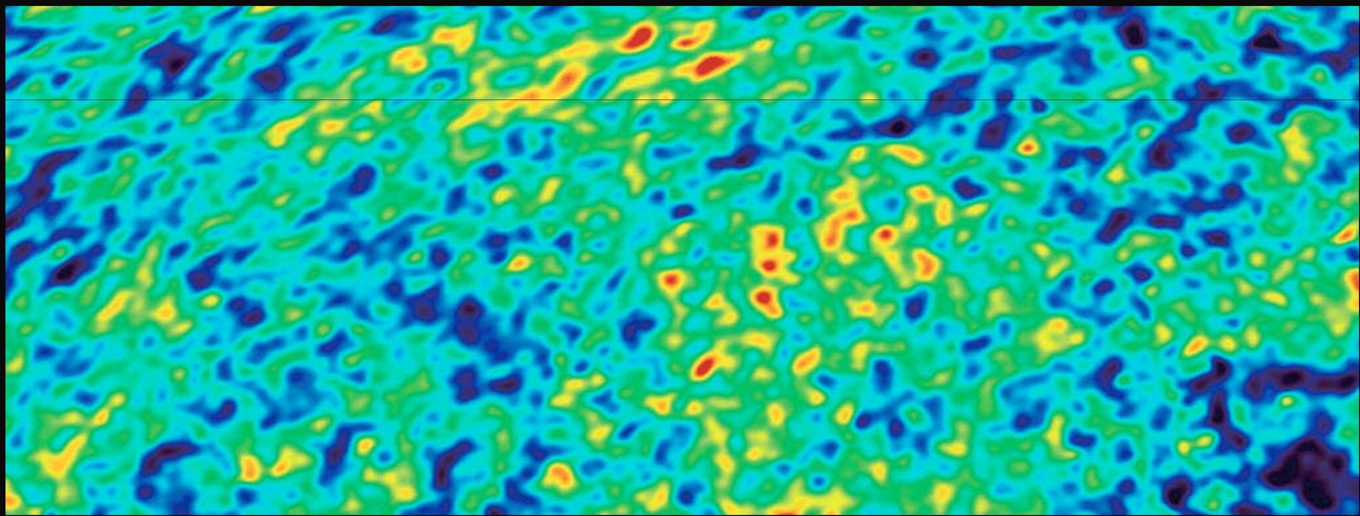
CMB data

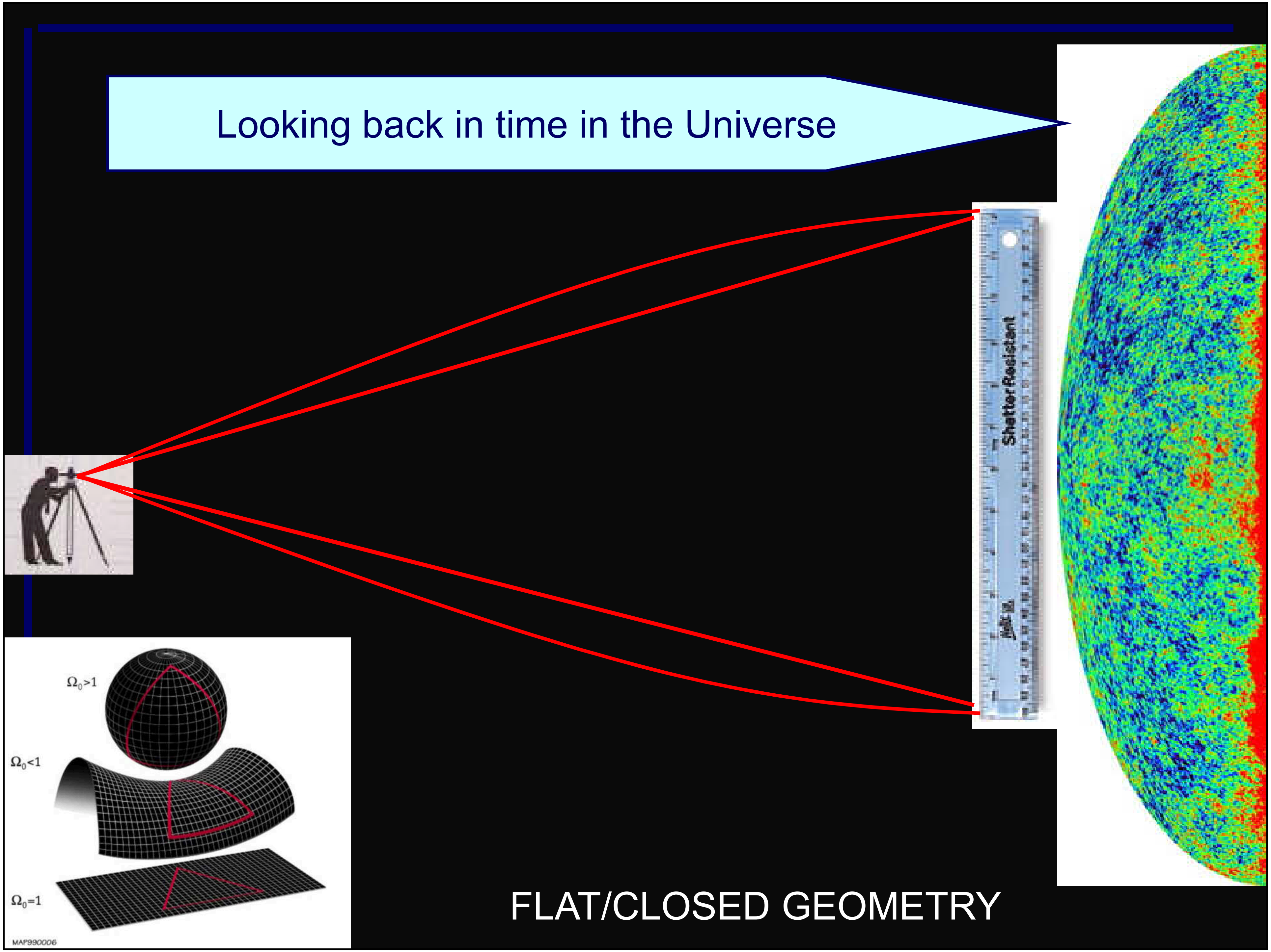
- Typical size of the blobs on the CMB?
  - 1 deg
  - Why?



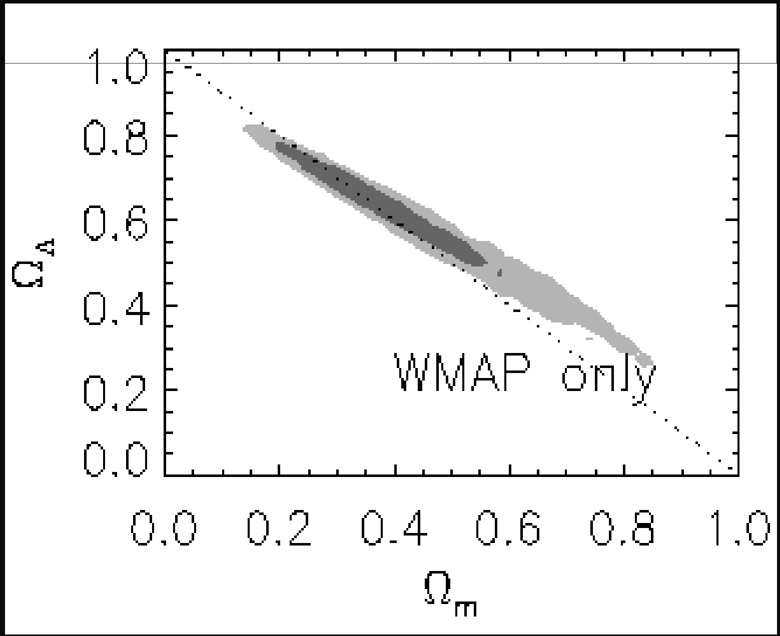
A characteristic scale exists of ~ 1 degree

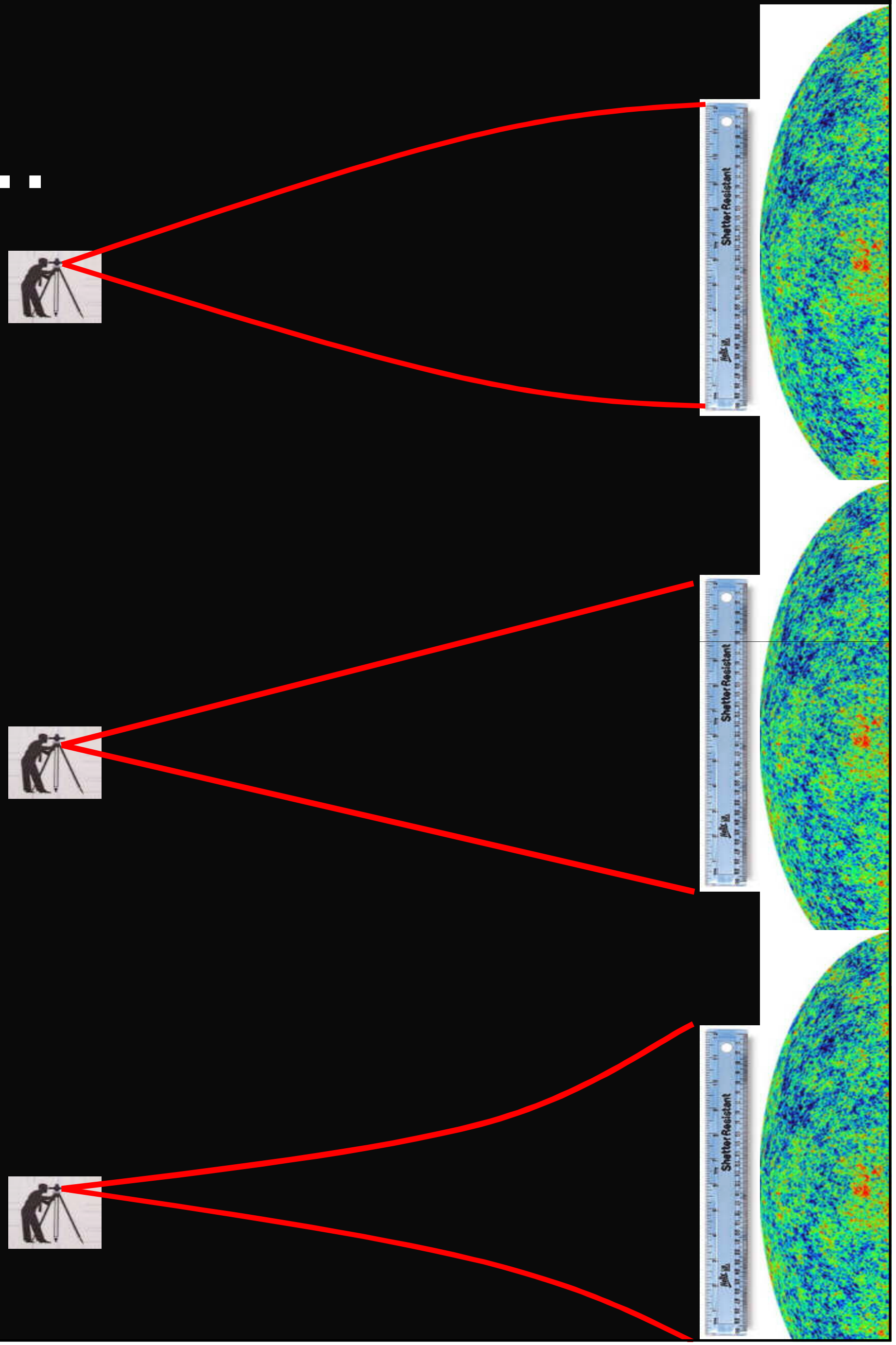




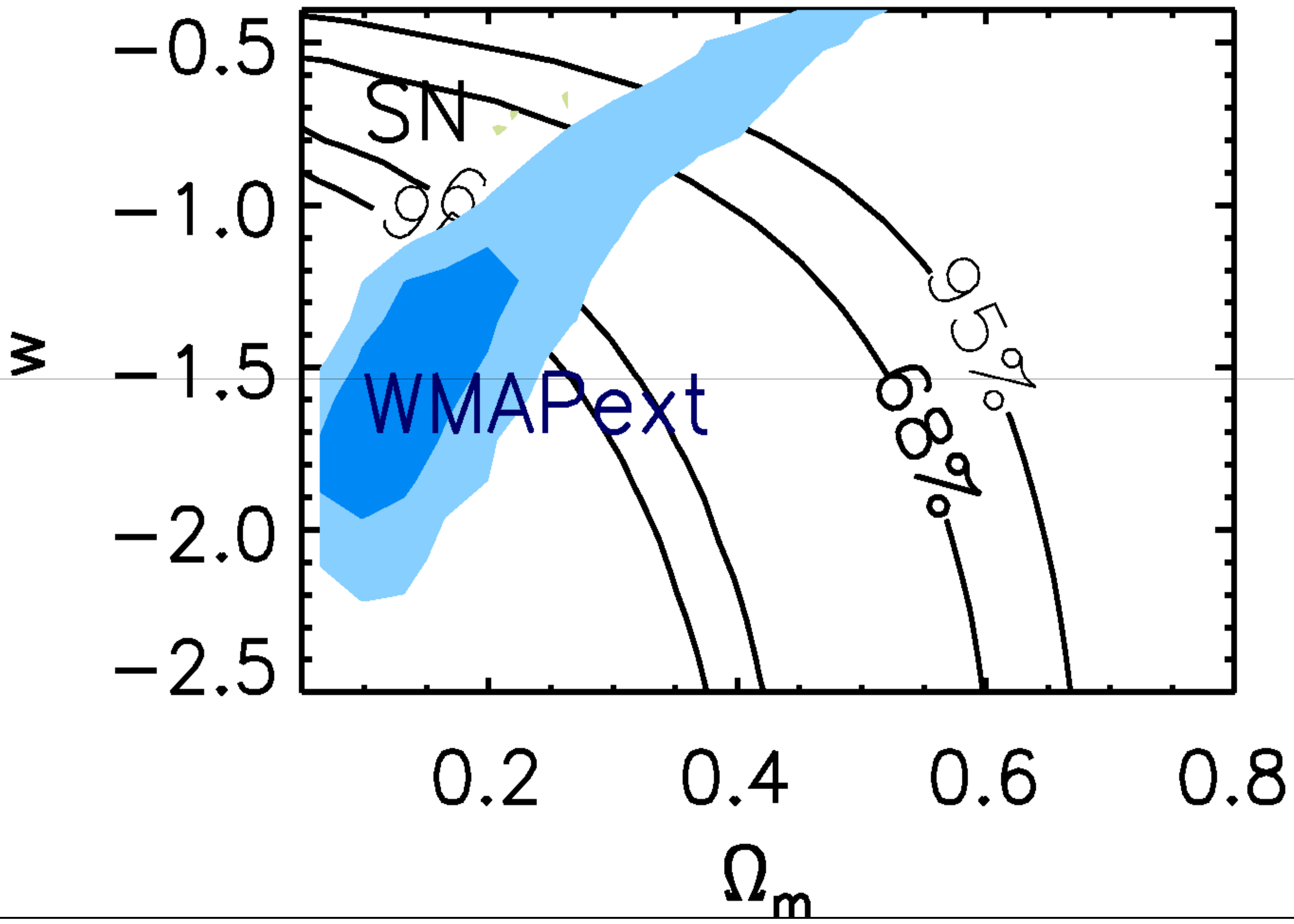

Looking back in time in the Universe
Shatter Resistant
Ω0>1
Ω0<1
Ω0=1
MAP990006
FLAT/CLOSED GEOMETRY

Joint constraints…





Current SN, if Ω_K=0, w(z)=constant



The Volume of the Universe

- If we take the volume element to be:

$$\mathrm{d}V = 4\pi r^2 \mathrm{d}r$$

- Or for a opened/closed Universe

$$\mathrm{d}V = 4\pi r^2 \frac{\mathrm{d}r}{\sqrt{1 - kr^2}}$$

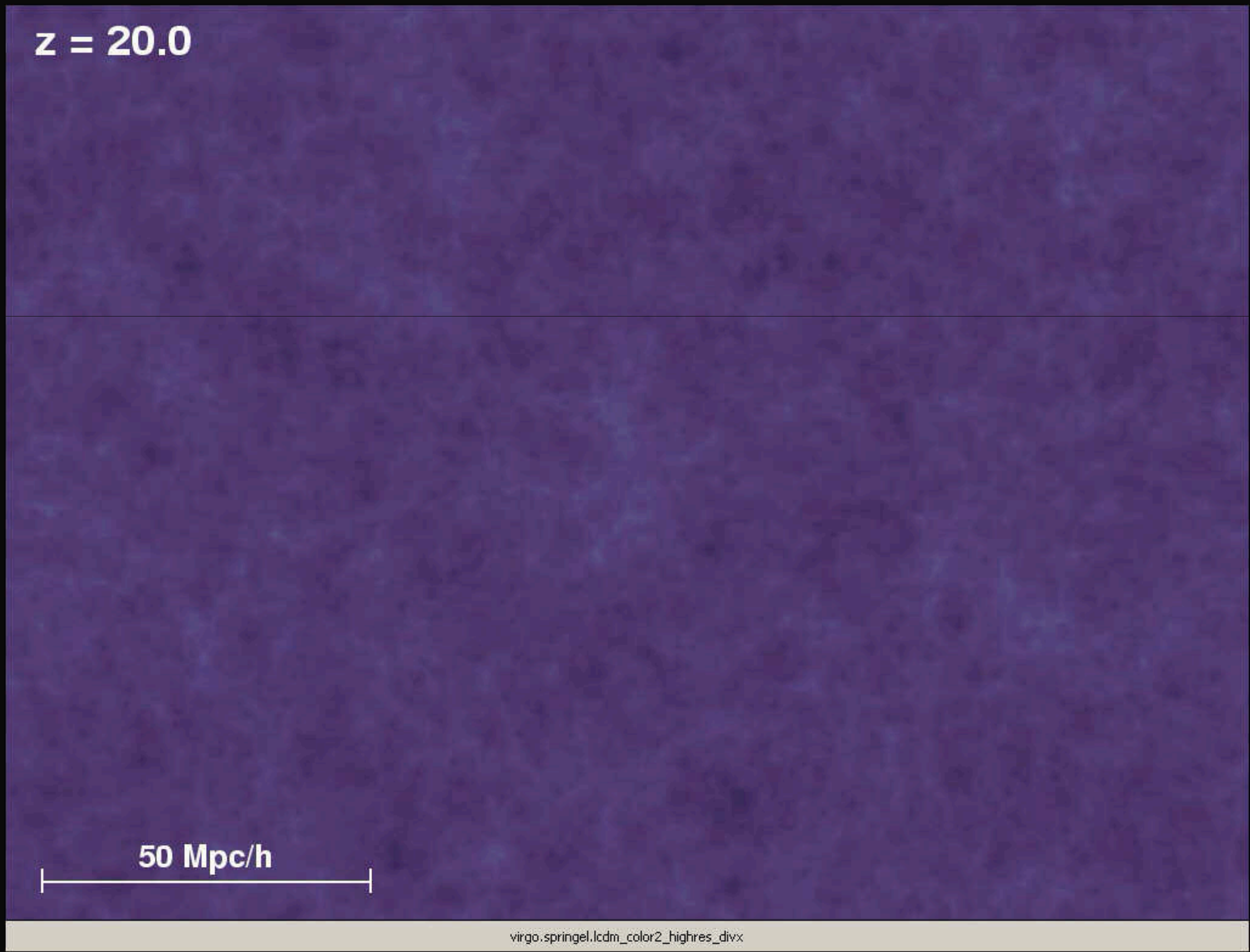
- The volume can be related to the Hubble factor. This allows us to count objects as a cosmological probe.

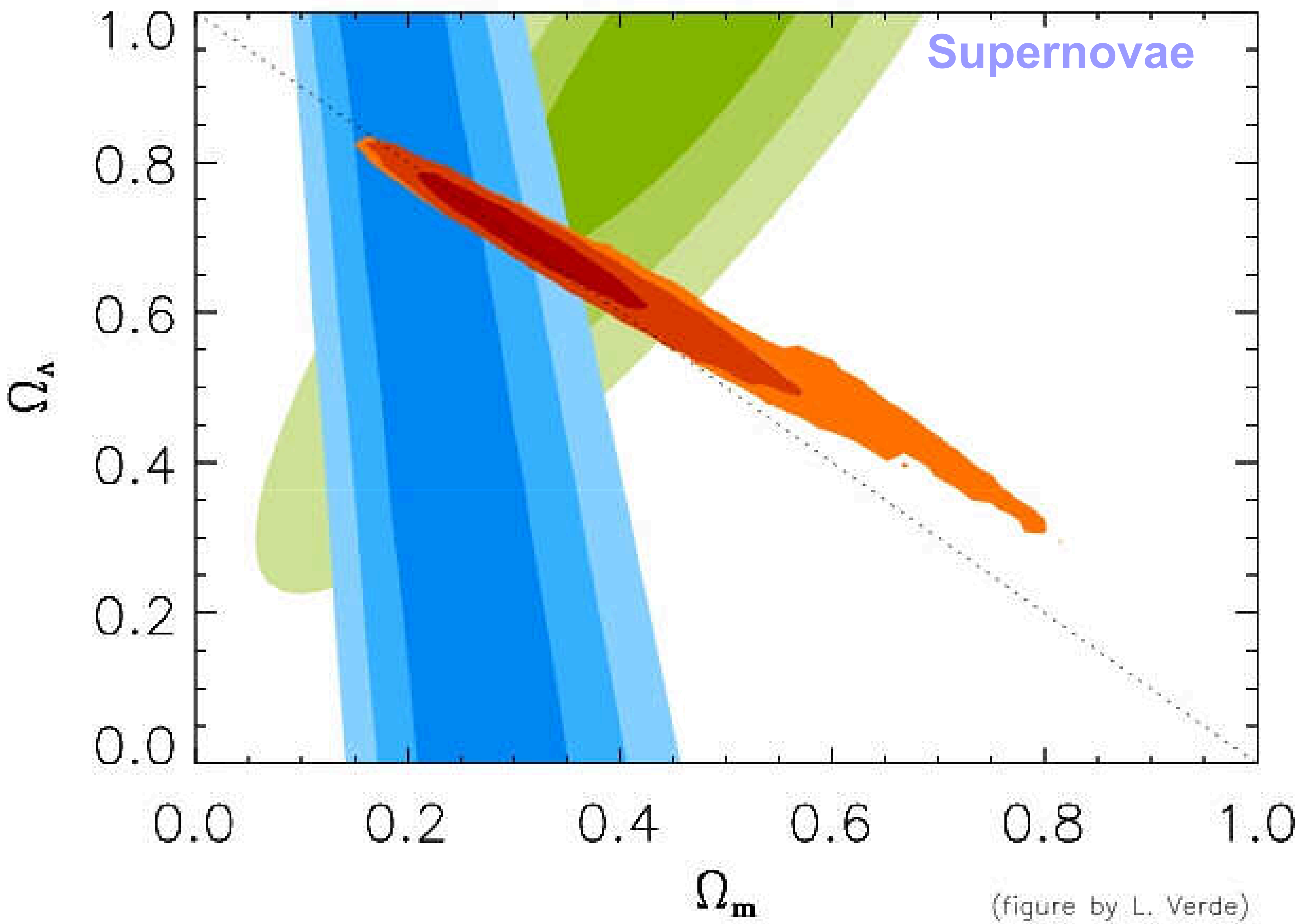
Clusters

- If we know how many clusters there are per unit volume we can use that as well to probe cosmology:
- The problem is how to fund the number of clusters per unit volume.
  - Press-Schechter (not examinable)
  - N-body simulations (not examinable)

$$\frac{\mathrm{d}N}{\mathrm{d}z} = \int\int \frac{\mathrm{d}^2 n}{\mathrm{d}V\mathrm{d}M}\frac{\mathrm{d}V}{\mathrm{d}z}\mathrm{d}z\mathrm{d}M$$

How to count the number of halos per volume: N-body simulations

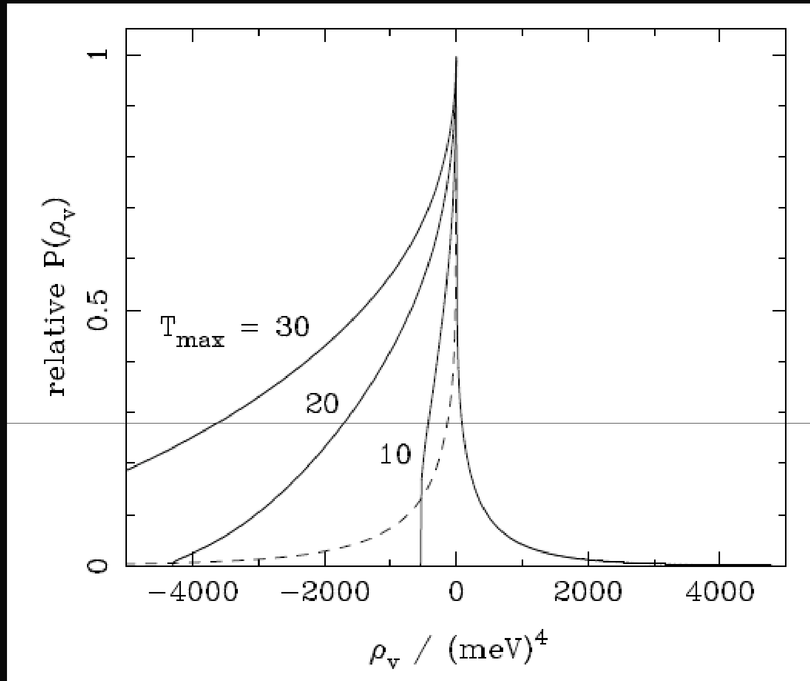




http://www.physics.upenn.edu/~lverde/cosmmodel.html

Back to the anthropic principle:

- Votes!
- Discussion!
- Is cosmology a science?



- Is the anthropic principle valid?
- Cosmology is a science based mainly on observations we can make from our statistical Universe…

END for now!!!