Observational cosmology: The Friedman equations 1

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A brain teaser: The anthropic principle!

- Last lecture I said “Is cosmology a science given that we only have one Universe?”
- **Weak anthropic principle:** "The observed values of all physical and cosmological quantities are not equally probable but they take on values restricted by the requirement that there exist sites where carbon-based life can evolve and by the requirements that the Universe be old enough for it to have already done so."
- **Strong anthropic principle**: "The Universe must have those properties which allow life to develop within it at some stage in its history.“
- Does it make sense to assume we exist and infer fundamental values for components of the Universe?
  - Can we say anything about Lambda given that we exist?
  - Can we say anything about the matter density given that we exist?
- If one could make a second Universe how do we know there would be life?

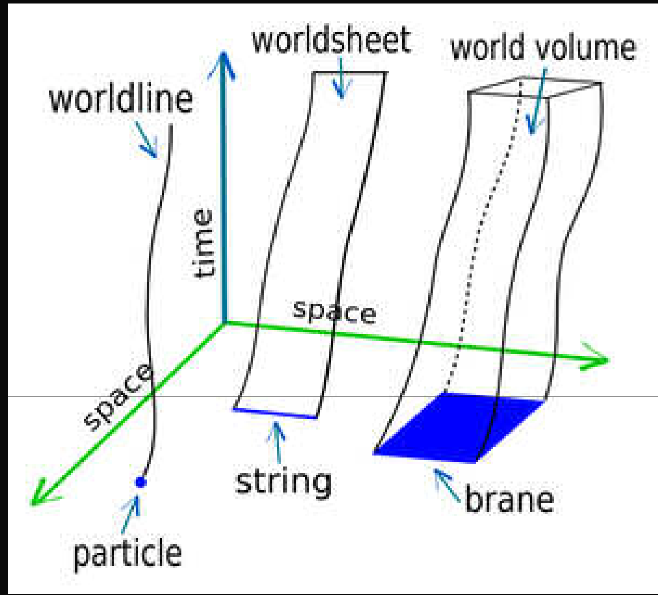
OUTLINE:
Friedman equation derivation

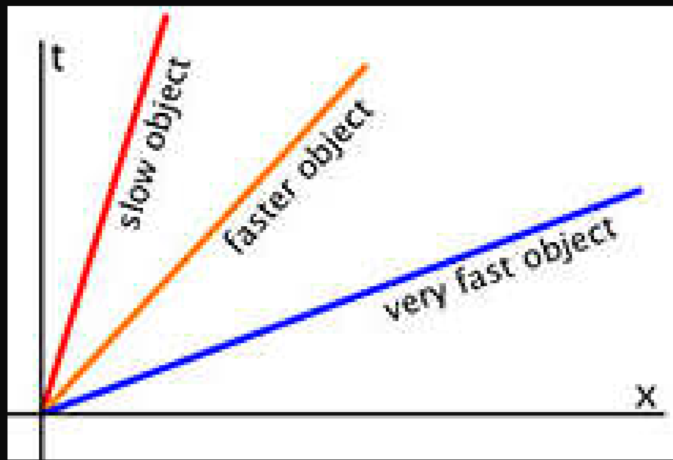
After the lecture, you should be able to:

- Define/derive the terms: “metric”, “scale factor” and “co-moving coordinates”
- Derive the relation between the scale factor and redshift
- Derive the Friedman equation using Newtonian arguments
- Describe and discuss the possible geometries of the Universe
- [Non-examinable: Discuss possible topologies of the Universe]

Our world line in special relativity!

- Is a one dimensional line or curve that represents the coordinates of a given place in space-time.
- As an objects moves the world line moves sideways. As time passes a static object moves along the z axis.
- Einstein said v<c. so world lines don’t bend more than 45 degrees or x/t > c.
- The photon world line defines boundaries of the knowable Universe from the unknowable Universe





The metric.

- In relativity space and time are mixed up so we have to define a distance which defines how separate 2 events are distant from one another in space-time.
- Infinitesimally separated events in space and time have a distance equal to:

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

- This is a space-time metric. It determines who one count distance between 2 points.
- If we measure distances from the origin there is no harm in choosing a spherical polar coordinate system:

$$ds^2 = c^2dt^2 - (dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2)$$

- So the proper time difference between two events is

$$\Delta\tau = \frac{\Delta s}{c}$$

The metric.

- For light: $\Delta\tau = \Delta s = 0$
- If $\Delta s < 0$

  the interval is a space like event
- If $\Delta s > 0$

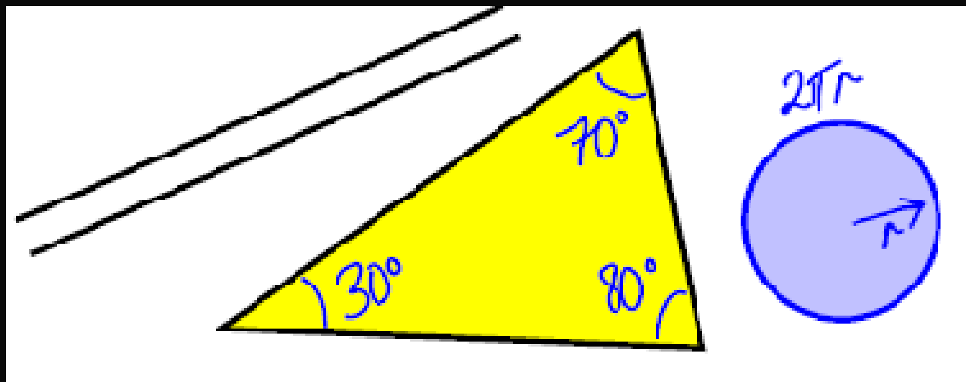
  the interval is a time like interval
- If space-time is curved then the metric defines the straightest possible world line. I.e. the geodesic. It is defined by:

$$\int \mathrm{d}s = 0$$

Euclidean geometry

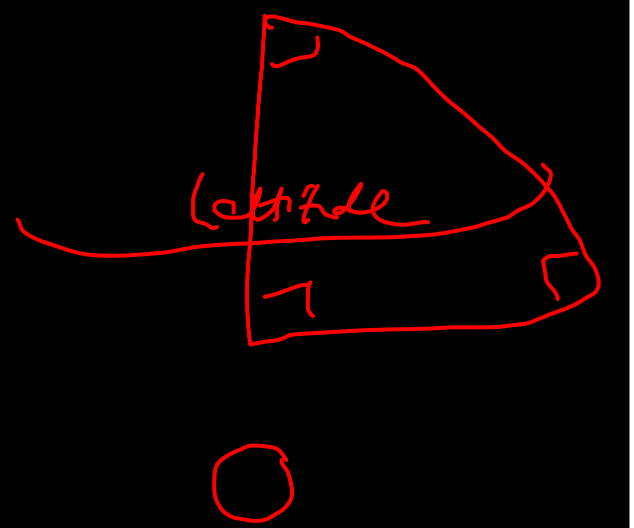
- = "flat" geometry

- parallel straight lines never meet
- triangle angles add up to 180°
- circumference of circle = 2π r
- NB. General definition of a straight line:
  - shortest distance between two points
  - applies for non-flat geometries



The surface of a objects…

- Are lines of longitude straight?
  - Yes, they are great circles
- Are lines of latitude straight?
  - No, they are not great circles
- Is the surface of a sphere flat?
  - No
- Is the surface of a cylinder flat?
  - Yes



The metric of a sphere

- The equation for a sphere: $$x^2 + y^2 + z^2 = a^2$$
- So if we calculate differentials $$2x\mathrm{d}x + 2y\mathrm{d}y + 2z\mathrm{d}z = 0$$
- This leads to $$\mathrm{d}z = -\frac{x\mathrm{d}x + y\mathrm{d}y}{z} = -\frac{x\mathrm{d}x + y\mathrm{d}y}{(a^2 - (x^2 + y^2))^{1/2}}$$
- If we go to a more convenient parameterisation $$\mathrm{x} = \rho\cos\phi$$ $$\mathrm{y} = \rho\sin\phi$$
- This means the space-like element is: $$\mathrm{d}l^2 = \frac{d\rho^2}{1 - (\rho/a)^2} + \rho^2\mathrm{d}\phi^2$$
- Notice the factor underneath the radial terms

The metric of a sphere embedded in 4D

- The equation for a 3 sphere embedded in 4D:

$$x^2 + y^2 + z^2 + w^2 = a^2$$

- So if we calculate differentials

$$2x\mathrm{d}x + 2y\mathrm{d}y + 2z\mathrm{d}z + 2w\mathrm{d}w = 0$$

- This leads to

$$\mathrm{d}w = -\frac{x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z}{(a^2 - (x^2 + y^2 + z^2))^{1/2}}$$

- If we go to a more convenient parameterisation

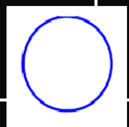
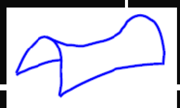
$$\mathrm{x} = r\sin\theta\cos\phi$$
$$\mathrm{y} = r\sin\theta\sin\phi$$
$$\mathrm{z} = r\cos\theta$$

- This means the space-like element is:

$$\mathrm{d}l^2 = \frac{dr^2}{1-(r/a)^2} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2$$

- Notice the factor underneath the radial terms … again!!!!!
  we define k = 1/a^2

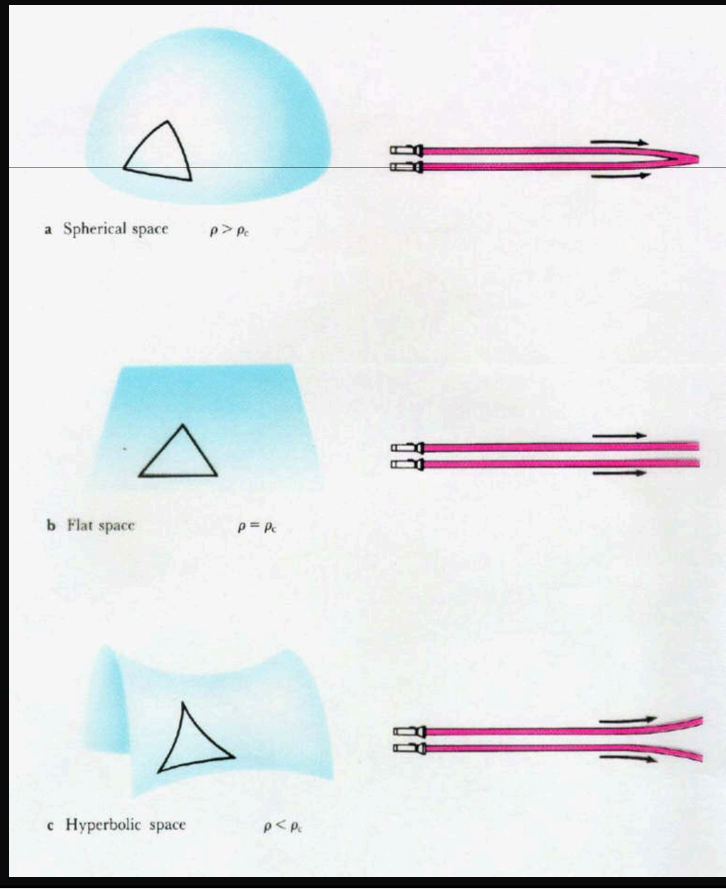
Summary of Geometries

Curvature	Geometry	Angles of triangle	Circumference of circle	Type of Universe
$k > 0$ 	spherical	$>180°$	$c < 2 \pi r$	Closed
$k = 0$ 	flat	$180°$	$c = 2 \pi r$	Flat
$k < 0$ 	hyperbolic	$<180°$	$c > 2 \pi r$	Open

- Copy of Liddle Table 4.2:

The meaning of k in $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$

- $k = 1/R^2$ (from GR), where R = radius of curvature
- Flat
  - R = infinity
  - $k = 0$
- Spherical
  - R < infinity
  - $k > 0$
- Hyperbolic
  - R imaginary
  - $k < 0$



http://www.dur.ac.uk/Physics/students/csflevel1/cffig1.jpg

The FRW metric:

- We want to be general… i.e. for our Universe we want to write a metric which is isotropic.

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - [f_1(r,t)\mathrm{d}r^2 + f_2(r,t)(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)]$$

- For a homogeneous and isotropic Universe we can prove that (from GR later…)

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - a(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\right]$$

- k can be associated with the Gaussian curvature of the Universe…
- Fro convenience we can change variables to:

$$\begin{array}{ll} r = (1/\sqrt{k})\sin(\sqrt{k}\chi) & k > 0 \\ r = \chi & k = 0 \\ r = (1/\sqrt{|k|})\sinh(\sqrt{|k|}\chi) & k < 0 \end{array}$$

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - a(t)\left[\mathrm{d}\chi^2 + S^2(\chi)(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\right]$$

The derivation of the redsfhit

- For a photon, using the FRW metric we have:

$$c^2\mathrm{d}t^2 = a^2(t)\frac{\mathrm{d}r^2}{(1-kr^2)}$$

- How if we take a photon arrival and emission time:

$$\frac{1}{c}\int_0^{r_e}\frac{\mathrm{d}r}{\sqrt{1-kr^2}} = \int_{t_e}^{t_o}\frac{\mathrm{d}t}{a(t)}$$

- The second equality is because we can take the second crest given that the photon is a wave.

$$\frac{1}{c}\int_0^{r_e}\frac{\mathrm{d}r}{\sqrt{1-kr^2}} = \int_{t_e+\Delta t_e}^{t_o+\Delta t_o}\frac{\mathrm{d}t}{a(t)}$$

- So equating the two we have

$$\int_{t_o}^{t_o+\Delta t_o}\frac{\mathrm{d}t}{a(t)} = \int_{t_e}^{t_e+\Delta t_e}\frac{\mathrm{d}t}{a(t)}$$

- If we assume a(t) is unchanging during these small intervals we can take a out of the integral.

- Hence:

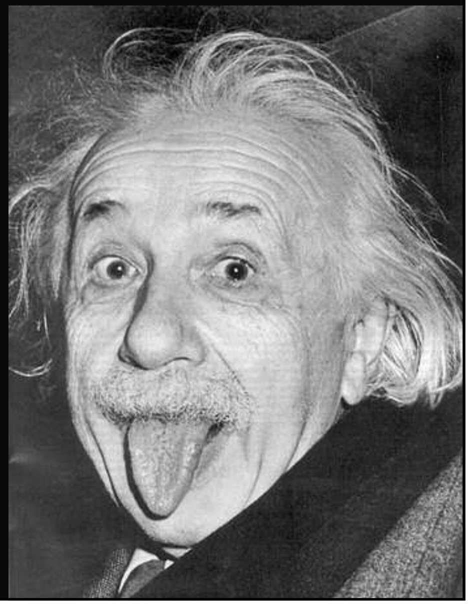
$$\frac{\Delta t_e}{a(t_e)} = \frac{\Delta t_o}{a(t_o)}$$

- So if we take the time intervals to be the periods of the photon we have now:

$$1+z = \frac{\lambda_0}{\lambda_e} = \frac{\Delta t_o}{\Delta t_e} = \frac{a_o}{a_e}$$

Principle of Equivalence:

- Constant ratio of the inertial mass and the gravitational mass
- Means that “All local freely falling, non-rotating labs are fully equivalent for the purposes of physical experiments”
- i.e. the strong equivalent principle t is always possible to choose a local co-ordinate system such that all the laws of physics have the same form.
- Acceleration = gravitation = curvature



Derive the Friedmann equation

- Describes expansion rate of Universe
- a = scale factor
- da/dt = differential of a wrt time
- ρ = matter density
- k = a constant

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

- Full derivation uses GR.
- Use Newtonian derivation here

Friedman eq. derivation:

- Lets follow Newton and write the force on a mass m.

$$F = \frac{GMm}{r^2} = \frac{4\pi G m \rho r}{3}$$

- The particle's gravitational potential can be written as:

$$V = -\frac{GMm}{r} = -\frac{4\pi G m \rho r^2}{3}$$

- The kinetic energy of a particle can be written as:

$$T = \frac{1}{2} m \dot{r}^2$$

- Energy conservations gives us:

$$U = T + V = \frac{1}{2} m \dot{r}^2 - \frac{4\pi G m \rho r^2}{3}$$

- The relation between the position and the co-moving position is:

$$r = a(t)x$$

- So the total internal energy of the system is:

$$U = \frac{1}{2} m \dot{a}^2 x^2 - \frac{4\pi}{3} G m \rho a^2 x^2$$

- Making the substitution:

$$kc^2 = -\frac{2U}{mx^2}$$

- We finally have: (we will prove that this is the Hubble constant later)

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G \rho}{3} - \frac{kc^2}{a^2}$$

General Relativity in two slides: Christophel, Ricci and Riemann (non examinable)

- All of Special Relativity applies.
- Worldlines are straight.
- In a non-inertial frame, there are accelerations:

$$\frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda{}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

where the Christophel symbol is:

$$\Gamma^\sigma{}_{\lambda\mu} = \frac{1}{2}g^{\nu\sigma}\left\{\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu}\right\}$$

and the Riemann tensor and Ricci tensor and scalars are defined as:

$$R^\alpha{}_{\sigma\rho\beta} \equiv \Gamma^\alpha{}_{\beta\sigma,\rho} - \Gamma^\alpha{}_{\rho\sigma,\beta} + \Gamma^\alpha{}_{\rho\nu}\Gamma^\nu{}_{\sigma\beta} - \Gamma^\alpha{}_{\beta\nu}\Gamma^\nu{}_{\sigma\rho}$$

$$R_{\alpha\beta} \equiv R^\mu{}_{\alpha\mu\beta}; \quad R \equiv R^\mu{}_\mu$$

General Relativity in two slides: Einstein………. (non examinable)

- Einstein told us that

$$G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu}$$

and also that

$$G^{\mu\nu} - \Lambda g^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu}$$

- For perfect fluids

$$T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) U^\mu U^\nu - \frac{p}{c^2} g^{\mu\nu}$$

where U is the 4 momentum, and these Einstein equations reduce to 2 Friedman equations cosmologists use…

$$\begin{aligned} \dot{R}^2 + kc^2 - \frac{\Lambda}{3} c^2 R^2 &= \frac{8\pi G\rho}{3} R^2 \\ 2\frac{\ddot{R}}{R} + \frac{\dot{R}^2 + kc^2}{R^2} - \Lambda c^2 &= -\frac{8\pi G p}{c^2} \end{aligned}$$

Implications of Friedmann eqn

- Allows us to solve for a(t)
- If we know ρ(t) and k
- ρ(t) depends on contents of Universe
  - see next lecture on fluid equation

$+\frac{\Lambda}{3}$

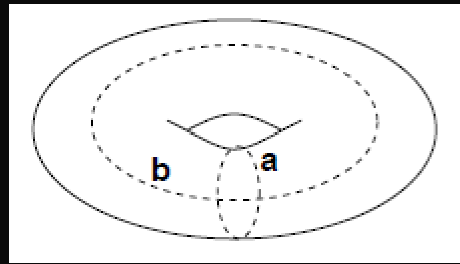
- What is the meaning of k?
  - From GR derivation
  - geometry of the Universe What is the meaning of k?
- In GR we can sa we want a maximally symetric metric only dependent on the curvature so

$$R_{ij} = -2K\, g_{ij}$$

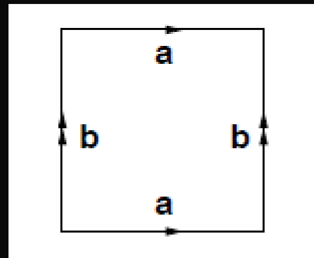
- Implies that the metric should be the way it was derived before

Topologies:

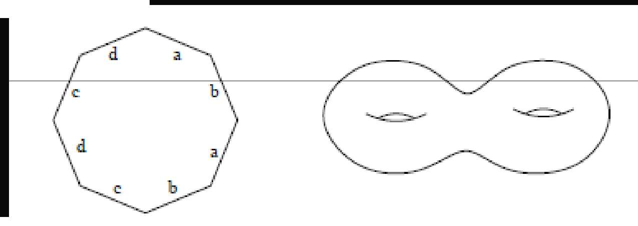
- So far we look at a patch of the Universe but is the Universe infinite or a tiling of bits?



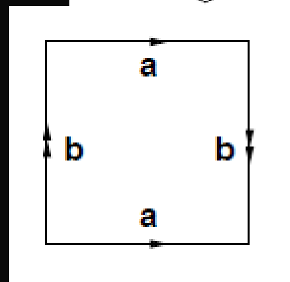
- One example:
  - What kind of Universe is this?
  - A Thorus:



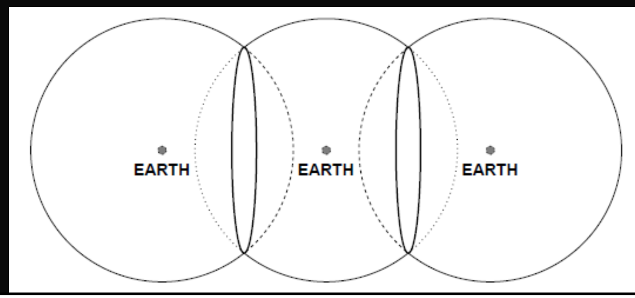
- Other examples: a bit weird but possible! Called a torus of genus 2



- Simpler but still unusual: what is this?



- One consequence: we can see ourselves in the past if we look far away enough!!!!!!! We can test this seeing circles in the CMB



OUTLINE
Acceleration equation derivation

After the lectures, you should be able to:

- Find the change in density as a function of scale factor $\rho(a)$ for matter dominated universe, just by considering conservation of matter
- Derive the fluid equation
- Derive the change in density as a function of scale factor $\rho(a)$ for a single fluid universe with given equation of state, from the fluid equation
- Derive the acceleration equation from the Friedmann equation and the fluid equation
- Discuss the cosmological constant and dark energy

The fluid equation

- The change of volume as a function of time can be written as:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 4\pi a^2 \frac{\mathrm{d}a}{\mathrm{d}t} x^3$$

- But if we write the energy as:

$$E = mc^2 = \frac{4\pi}{3} a^3 x^3 \rho c^2$$

- So the change in energy as a function of time is:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = 4\pi a^2 x^3 \rho c^2 \frac{\mathrm{d}a}{\mathrm{d}t} + \frac{4\pi}{3} a^3 x^3 \frac{\mathrm{d}\rho}{\mathrm{d}t} c^2$$

- Here we start with the second law of thermodynamics:

$$\mathrm{d}E = T\mathrm{d}S - p\mathrm{d}V$$

- So we have if we assume the expansion is reversible, i.e. isentropic:

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0$$

The acceleration equation

- We take the Friedman equation in the following form and differentiate it:

$$\dot{a}^2 = \frac{8\pi G\rho}{3}a^2 - kc^2$$

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3}(\dot{\rho}a^2 + 2\rho a\dot{a})$$

- Now we substitute the value of the differential of the density from the fluid equation back into this equation to get:

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0$$

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3}(-3\dot{a}a(\rho + \frac{p}{c^2}) + 2\rho a\dot{a})$$

- Simplifying we get:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3\frac{p}{c^2}\right)$$

Derivation for the density evolution for a component with w.

- We can re-write the acceleration equation and the fluid equation with the natural parameter w the ratio of the pressure to the energy density.
- Re-writing the fluid equation in such way:
- We can describe how different components with different w evolve as a function of the scale factor.
- So for example we have proven that if a component such as matter has no pressure, then the density varies as the cubed power of a

$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho(1+w) = 0$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho(1+3w)$$

$$\frac{\mathrm{d}\rho}{\rho} = -3\frac{\mathrm{d}a}{a}(1+w)$$

$$\rho = \rho_0 \exp\left(-3\int \frac{\mathrm{d}a}{a}(1+w(a))\right)$$

Different component evolutions:

- Matter: w = 0 varies as a cubed
- Radiation w = 1/3 varies as a to the 4
- A cosmological constant is has rho constant so w must be -1.
- Any particle which goes from relativistic to non-relativistic has w from 1/3 to 0.

$$\rho = \rho_0 \exp\left(-3\int \frac{\mathrm{d}a}{a}(1 + w(a))\right)$$

Matter/radiation domination

- In a matter dominated universe, given that the density goes as the cube of the scale factor:

$$\dot{a}^2 = \frac{8\pi G\rho}{3} a^2 = \frac{8\pi G\rho_0}{3}\frac{1}{a}$$

- Try a solution

$$a \propto t^q$$

- For matter domination we have: q=2/3.

$$a = (t/t_0)^{(2/3)}$$

- So the Hubble parameter can be written as :

$$H(t) = \frac{2}{3t}$$

- Implications for the age of the Universe! This value is less than the age of some systems. Which?
- For radiation domination q=1/2

$$a = (t/t_0)^{(1/2)}$$

The cosmological constant

- Einstein spotted a constant of integration
- Appears in our equations:

$$G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}$$

$$\frac{\ddot{a}}{a} = -4\pi G(\rho + 3p) + \frac{\Lambda}{3}$$

- Out of fashion (assumed zero) until ~ 10 years ago
- Looks just like a fluid with w=-1
  - has negative pressure!
- The vacuum energy from particle physics could produce this effect
  - current calculations give Λ a factor 10^{120} too high!
  - Compare the energy in dark energy to GUT scale energies…
- Implies $\rho(a)$ = constant, despite expansion of the Universe!

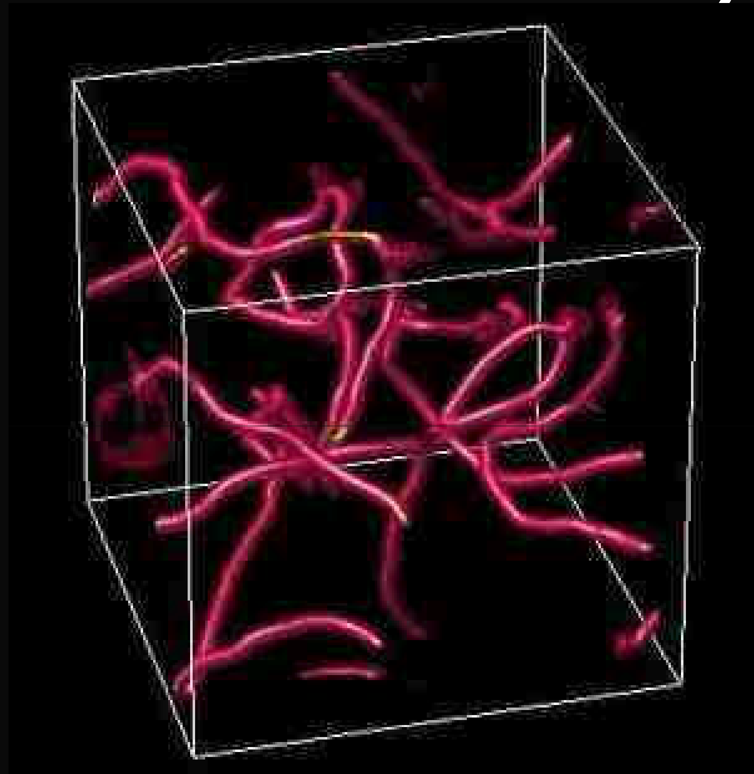
It is by looking at the second equation that Supernovae people have told us the universe is accelerating!!!!!

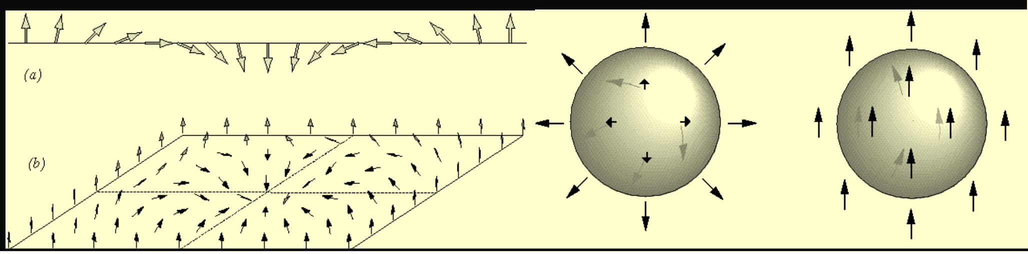
- Acceleration: $$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\left(\rho + 3\frac{p}{c^2}\right) + \frac{\Lambda c^2}{3}$$
- Important parameter is: $w = p/\rho c^2$
- For acceleration we need $w < -1/3$
- A Network of cosmic strings has w = -1/3, first check more than a decade ago!

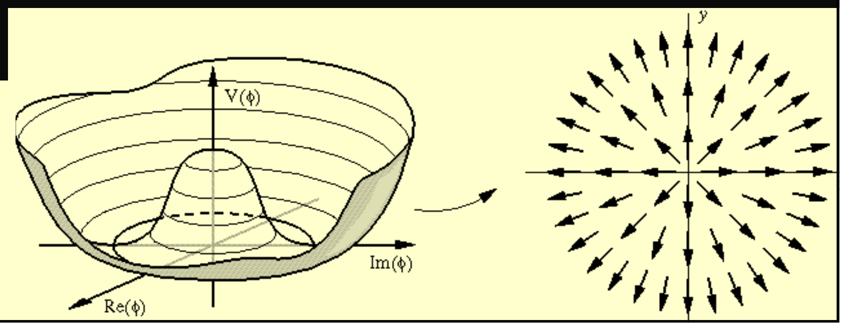
BTW here we are not assuming a lambda, it COULD be there or a term with w also COULD be there…

Network of cosmic strings and topological defects: (not examinable)

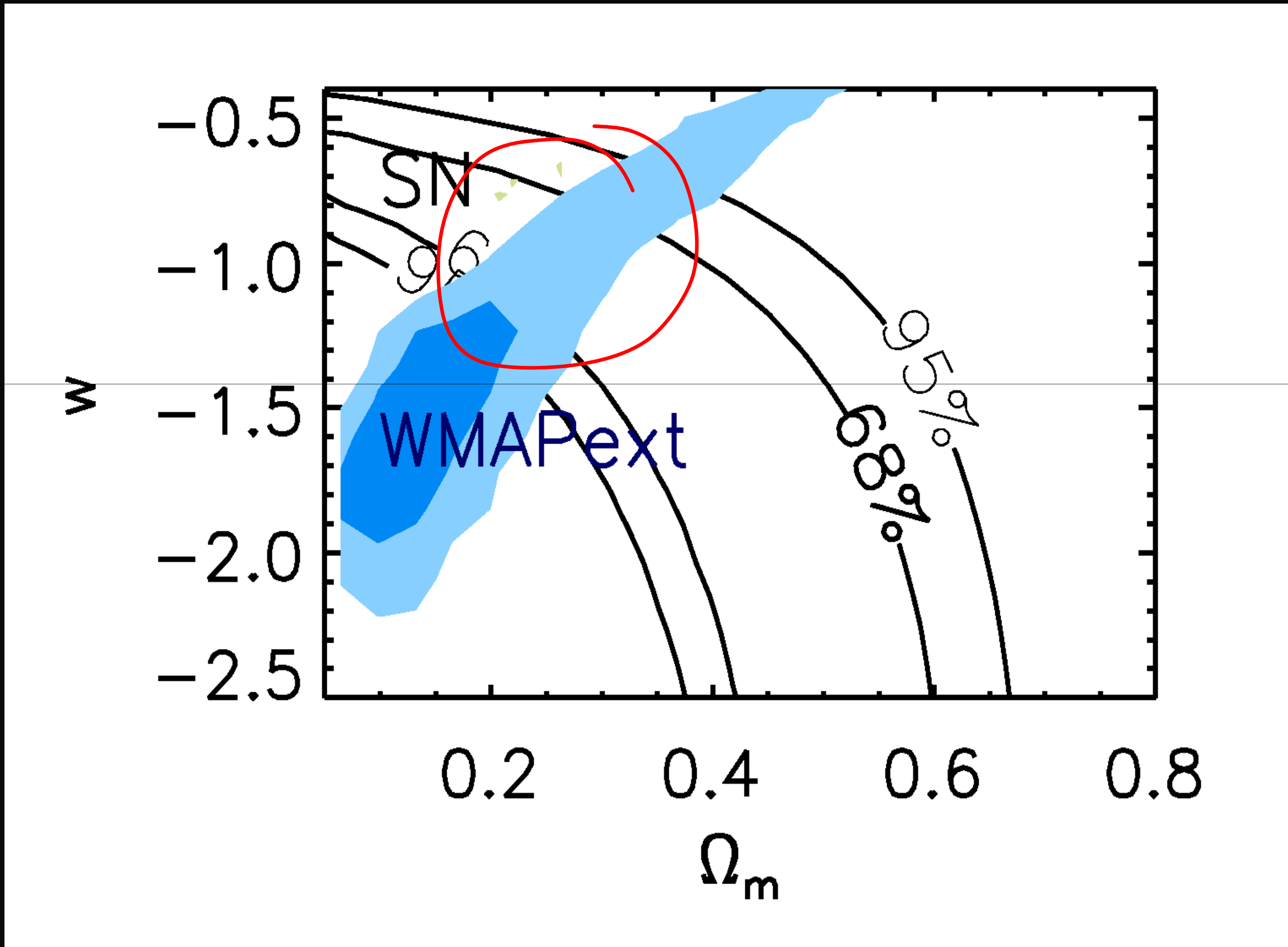
- A configuration formed at a phase transition in the very early Universe:
- Can be:
  - Monopoles
  - Strings
  - Domain walls
  - Textures (non localised unstable)
- A mechanism for forming these objects is called the Kibble mechanism
- Could explain accelerated expansion but…. Power spectrum very different (we will see this…)







Constraints from the CMB (blue) cf Supernove



WMAP team

Spergel et al 2003

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END for now!!!