Simulation Rescalings

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March 24, 2006

1 Introduction

In studying cosmological dependence of, for example, clustering statistics, it is convenient to have N-body simulations in a variety of cosmologies. Then mock galaxy catalogues etc. can be constructed for these different cosmologies and their properties studied.

We describe here some methods by which a simulation in one cosmology can be used to mimic a simulation in a different cosmology sufficiently well that the errors are not significant in the context of properties of the mock catalogues. Fewer computationally expensive N-body simulations are then required, which may be a considerable saving if the simulations are large. Where relevant, we concentrate on rescaling output from GADGET (Springel et al., 2001).

2 Preliminaries

Let comoving distances be denoted by x, and physical distances be denoted by $r = ax$, so that a is the conventional scale factor, normalised such that its value at redshift zero is $a_0 = 1$. Let the Hubble Parameter $H(a) = H_0 h(a)$, where $H_0 = 100$ km s^{-1} Mpc^{-1} and the present value of the Hubble Parameter is $H(1) = H_0 h_0$, that is $h_0 = h(1)$.

Now we may write the critical density

$$\rho_{\rm crit}(a) = \frac{3H_0^2 h^2(a)}{8\pi G} \quad . \tag{1}$$

By definition, we have $\rho = \rho_{\rm crit}\Omega$. Below, I will consider ρ without a subscript to be the matter density and Ω to be the ratio of matter density to critical density. A subscript zero denotes their values at redshift zero as is conventional. Then,

$$\rho(a) = \frac{\rho_0}{a^3} = \rho_{\rm crit}(a)\Omega(a) = \frac{3H_0^2}{8\pi G} h^2(a)\Omega(a) \quad , \tag{2}$$

that is,

$$h^2(a)a^3\Omega(a) = \frac{8\pi G}{3H_0^2}\rho_0 \tag{3}$$

$$= h_0^2\frac{8\pi G}{3H_0^2h_0^2}\rho_0 \tag{4}$$

$$= h_0^2\Omega_0 \tag{5}$$

which is a constant in any given cosmology.

Consider an isolated halo in its centre-of-mass frame in this cosmology, consisting of particles labelled by $i = 1, 2, \ldots, n$. Its potential energy is given by

$$E_{\mathrm{P}} = -\sum_i\sum_{j>i}\frac{Gm_im_j}{|r_i - r_j|} \tag{6}$$

$$= -\frac{1}{2a}\sum_i\sum_{j\neq i}\frac{Gm_im_j}{|x_i - x_j|} \quad , \tag{7}$$

where we treat the positions as scalar quantities for clarity, since the generalisation to three dimensions is trivial. Its kinetic energy is then given by

$$E_{\mathrm{K}} = \frac{1}{2}\sum_i m_i\left(\frac{\mathrm{d}r_i}{\mathrm{d}t}\right)^2 \tag{8}$$

$$= \frac{1}{2}\dot{a}^2\sum_i m_i\left(x_i + a\frac{\mathrm{d}x_i}{\mathrm{d}a}\right)^2 \tag{9}$$

$$= \frac{1}{2}a^2H_0^2h^2(a)\sum_i m_i\left(x_i + a\frac{\mathrm{d}x_i}{\mathrm{d}a}\right)^2 \tag{10}$$

after some elementary manipulation.

There is a slight complication, in that it is conventional to use a system of units in which

$$m_i = \frac{\widetilde{m}_i}{h_0} \quad \text{and} \quad x_i = \frac{\tilde{x}_i}{h_0} = ar_i = \frac{a\tilde{r}_i}{h_0} \quad . \tag{11}$$

In this system, the energies are also scaled in the same way, $\widetilde{E} = h_0E$. Then,

$$\widetilde{E}_P = \frac{-1}{2a}\sum_i\sum_{j\neq i}\frac{G\widetilde{m}_i\widetilde{m}_j}{|\tilde{x}_i - \tilde{x}_j|} \tag{12}$$

and

$$\widetilde{E}_K = \frac{1}{2}\left(\frac{aH_0h(a)}{h_0}\right)^2\sum_i\widetilde{m}_i\left(\tilde{x}_i + a\frac{\mathrm{d}\tilde{x}_i}{\mathrm{d}a}\right)^2 \quad . \tag{13}$$

GADGET stores the particle positions $\tilde{x}$, using units of h_0^{-1} Mpc. Storing the velocities is slightly more problematic. In fact, the quantity stored is

$$w = a^{1/2} H_0 h(a) a \frac{\mathrm{d}x}{\mathrm{d}a} \quad , \tag{14}$$

which corresponds to peculiar, physical velocities in km s^{-1}, divided by $a^{1/2}$. The factor of $a^{1/2}$ is introduced for numerical convenience — velocities are expected to scale with $a^{1/2}$ by linear theory, so dividing by this factor ensures that velocities do not change by several orders of magnitude during the calculation which could introduce numerical errors. We can see this correspondence between w and peculiar, physical velocities by observing that

$$\begin{aligned} \frac{\mathrm{d}r}{\mathrm{d}t} &= \frac{\mathrm{d}(ax)}{\mathrm{d}t} && (15) \\ &= \dot{a}x + a\frac{\mathrm{d}x}{\mathrm{d}t} && (16) \\ &= \dot{a}x + a\dot{a}\frac{\mathrm{d}x}{\mathrm{d}a} && (17) \\ &= \dot{a}x + a^2 H(a)\frac{\mathrm{d}x}{\mathrm{d}a} && (18) \\ &= \dot{a}x + a^{1/2} w && (19) \end{aligned}$$

where the $\dot{a}x$ term corresponds to the Hubble flow.

So, we may now write the expressions to compute the kinetic and potential energy of our isolated halo in terms of the quantities w, $\tilde{x}$ and $\widetilde{m}$, where w and $\tilde{x}$ are as above and $\widetilde{m}$ is in units of h_0^{-1} M$_\odot$, such that the energies are numerically in units of h_0^{-1} M$_\odot$ (100 km s^{-1})^2 :

$$\widetilde{E}_P = -\frac{1}{a}\sum_i \sum_{j>i} \frac{G\widetilde{m}_i \widetilde{m}_j}{|\tilde{x}_i - \tilde{x}_j|} \quad ; \tag{20}$$

$$\widetilde{E}_K = \frac{1}{2}\sum_i \widetilde{m}_i \left(\frac{h(a)}{h_0} a\tilde{x}_i + \frac{a^{1/2} w}{H_0} \right)^2 \quad . \tag{21}$$

Note that here $G = 4.301 \times 10^{-13}$ M$_\odot^{-1}$ Mpc (100 km s^{-1})^2.

These expressions will be useful in checking the validity of rescalings. The virial theorem indicates that, for an isolated halo in equilibrium,

$$\frac{2\widetilde{E}_K}{|\widetilde{E}_P|} \approx 1 \quad , \tag{22}$$

so that an analytic check on a rescaling of particle positions, velocities and masses is that the ratio of the kinetic and potential energies of halos is preserved. A numerical check on our results is that (22) holds before and after rescaling.

3 Simple Relabelling

Possibly the simplest useful rescaling to consider is one in which a $z > 0$ output with scale factor $a_{\rm i}$ from some simulation is treated as a $z = 0$ output from a simulation in a different cosmology. We wish the scale factor, a, to retain the conventional normalisation, i.e. $a_0 = 1$. So we must have $a \to 1$ under rescaling. Since the value of h_0 in the original simulation was presumably chosen to agree with available data, we also require $h_0 \to h_0$. Similarly, the peak in the initial power spectrum of density fluctuations will change unless $x \to x$. Then inspection of, e.g., (9) suggests that we also require

$$a\frac{{\rm d}x}{{\rm d}a} \to a\frac{{\rm d}x}{{\rm d}a} \quad . \tag{23}$$

This does not mean that the transformation is entirely trivial. If $\Omega \neq 1$ then the matter density in a simulation is a function of a, $\Omega \equiv \Omega(a)$. Since the point of doing the rescaling is that we acquire a simulation in a different cosmology, we preserve the value of Ω from the output we have rescaled; this becomes the value of Ω_0 in the new file. In other words, $\Omega_0 \to \Omega(a_{\rm i})$. For a flat, ΛCDM cosmology, this means the rescaled simulation has a higher Ω_0 than the old simulation. Since we have preserved our length scales, this implies that the particle mass must scale as

$$m \to \frac{\Omega(a_{\rm i})}{\Omega_0} m \quad , \tag{24}$$

where here Ω_0 refers to the $z = 0$ matter density before rescaling. Since (23) implies that $a^{-1/2} w/h(a)$ is also preserved, we require

$$w \to a_{\rm i}^{-1/2} \frac{h_0}{h(a_{\rm i})} w = a_{\rm i} \left(\frac{\Omega(a_{\rm i})}{\Omega_0}\right)^{\frac{1}{2}} w \quad , \tag{25}$$

where we have used (5) to infer the equality on the right-hand side.

It remains to check that the kinetic and potential energies in an isolated halo scale the same way under this scheme. Substituting the new quantities into (20) and (21), and using (5) again, reveals that

$$E_K \to a_{\rm i} \left(\frac{\Omega(a_{\rm i})}{\Omega_0}\right)^2 E_K \quad \text{and} \quad E_P \to a_{\rm i} \left(\frac{\Omega(a_{\rm i})}{\Omega_0}\right)^2 E_P \quad , \tag{26}$$

so that we do indeed have consistency. Note that since h_0 is preserved, $\{m, x, E\}$ scale in the same way as $\{\widetilde{m}, \tilde{x}, \widetilde{E}\}$. Therefore the above gives us directly a prescription for how to alter the data in a GADGET output file to achieve the desired rescaling, bearing in mind that the output at any time contains the values of Ω_0 and h_0 — the values of $\Omega(a)$ and $h(a)$ at the final time — and not their instantaneous values at the output time. It is also worth noting that it is conventional to label a simulation by the value

of σ_8 — the scale of mass fluctuations in spheres of 8 h^{-1} Mpc according to linear theory — at the final time. However, this quantity also evolves with time, $\sigma_8 \equiv \sigma_8(a)$, becoming larger as structure forms in the simulation. So, $\sigma_8(z=0) \to \sigma_8(a_\mathrm{i})$, where $\sigma_8(a_\mathrm{i})$ can be calculated in linear theory.

4 More General Relabelling

Suppose that instead of relabelling the $a = a_\mathrm{i}$ output as an $a = 1$ output we decide to relabel the $a = a_\mathrm{i}$ output as an $a = a_\mathrm{f}$ output, for some $a_\mathrm{f} \neq a_\mathrm{i}$. A little more care in notation is required, but the transformations generalise as follows, with primes denoting quantities in the rescaled output:

$$
\begin{aligned}
a &\to a' && (27)\\
h_0 &\to h_0' = h_0 && (28)\\
h(a) &\to h'(a') \quad \text{such that } h_0' = h_0 && (29)\\
x &\to x' = x && (30)\\
a\frac{\mathrm{d}x}{\mathrm{d}a} &\to a'\frac{\mathrm{d}x'}{\mathrm{d}a'} = a\frac{\mathrm{d}x}{\mathrm{d}a} && (31)\\
\Omega(a) &\to \Omega'(a') = \Omega(a) && (32)\\
\Omega_0 &\to \Omega_0' \quad \text{such that the above holds} && (33)\\
w &\to w' = \frac{h'(a')}{h(a)}\left(\frac{a'}{a}\right)^{\frac{1}{2}} w && (34)\\
&\qquad = \left(\frac{\Omega_0'}{\Omega_0}\right)^{\frac{1}{2}} \frac{a}{a'} w \quad \text{using (5)} && (35)\\
m &\to m' = \frac{\Omega_0'}{\Omega_0} m \quad . && (36)
\end{aligned}
$$

Then it is easily checked, again using (5), that the kinetic and potential energies scale as

$$E \to E' = \frac{a}{a'}\left(\frac{\Omega_0'}{\Omega_0}\right)^2 E \quad . \tag{37}$$

5 Rescaling Ω

It has been observed that rescaling some parameters in cosmological simulations results, at least to first order, in a relatively straightforward scaling of some observables. For example, Zheng et al. (2002) state:

> For fixed linear theory $P(k)$, the effect of changing Ω_m is simple: the halo mass scale M_* shifts in proportion to Ω_m, pairwise velocities (at fixed M/M_*) are proportional to $\Omega_\mathrm{m}^{0.6}$, and halo clustering at fixed M/M_* is unchanged.

While this is an empirical effect rather than a consequence of an analytic calculation, we hope that the scaling is good enough so that our catalogues will still be sufficiently accurate when we rescale our simulations such that σ_8 remains constant but Ω_0 changes.

For example, suppose we choose our simulation such that one output has some fiducial values of Ω_0 and σ_8, once we take care of the relabelling described above. Presumably these values have been chosen to agree at some level with observations. Note, now, that Ω_0 and σ_8 may be constrained via some function of both parameters, for example using the observed abundance of clusters (Eke et al., 1996). It seems sensible, then, to generate an ensemble of catalogues such that they are all consistent with a cluster normalisation condition, where the curve of allowed values of (Ω_0, σ_8) passes through the fiducial point. This curve is not the same as the curve traced out by (Ω_0, σ_8) as the dark matter distribution in the simulation evolves. Therefore to generate our ensemble of catalogues we need not only to relabel the simulation outputs according to our analytic scheme above, but also to rescale the outputs (preferably by a small amount) so that the members of the ensemble lie on a convenient grid or on a suitable normalisation curve.

It is convenient to achieve this rescaling in practice by altering the particle mass to change Ω_0, and compensating by changing particle velocities. So,

$$\Omega_0 \rightarrow \Omega_0' \tag{38}$$

$$m \rightarrow m' = \frac{\Omega_0'}{\Omega_0} m \tag{39}$$

$$w \rightarrow w' = \left(\frac{\Omega_0'}{\Omega_0}\right)^{0.6} w \tag{40}$$

Note that although we would require w to scale as $(\Omega_0'/\Omega_0)^{0.5}$ to maintain the virial relation, the scaling we use is close to the linear theory prediction, and that in practice we only ever intend to rescale by small amounts. This is helped by the fact that the cluster normalization curve and the curve describing the evolution of the simulation parameters look qualitatively similar close to the fiducial point if we choose sensible parameter values.

References

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