Review of arXiv:0903.2208 (‘Principle of Maximum Entropy: the Origin of the Universality of Dark Matter Halo Density Profiles’, by Ping He)

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Principle of Maximum Entropy: the Origin of the Universality of Dark Matter Halo Density Profiles

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An isolated dark matter halo should satisfy the following three physical conditions at its stable or equilibrium state: (1) mass conservation, (2) energy conservation, and (3) virialization. Based on the principle of maximum entropy, with these three simple physical conditions as constraints, we successfully explained the origin of NFW density profile for dark matter halos. For this purpose, we exploited Rayleigh-Ritz method, which is the widely used method of direct variational calculus, to transform the functional extremal issue into the minimization of a multi-variable function. Then we used the standard augmented Lagrangian method to perform the numerical optimization. The density profile is theoretically confirmed to be universal, in that they are irrelevant to the halo mass, radius, red-shift, cosmological parameters, and their histories of formation and evolution. Furthermore, the universality can even be generalized to the cases of other dark matter variants, such as hot dark matter, warm dark matter, collisional dark matter, and so forth.

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The NFW profile

- Density profile: $$\frac{\rho(r)}{\rho_{\rm crit}} = \frac{\delta_{\rm c}}{(r/r_{\rm s})(1+r/r_{\rm s})^2}$$
- r_s is a scale radius, roughly corresponding to where the profile is isothermal.
- δ_c is a characteristic density related to c=r_s/r_{200} by:

$$\delta_{\rm c} = \frac{200}{3}\frac{c^3}{\ln(1+c) - c/(1+c)}$$

- δ_c related to the density of the Universe at the redshift of collapse:

$$\delta_{\rm c}(M|f) = C\Omega_0[1+z_{\rm coll}(M,f)]^3$$

- Density profile implies a velocity profile:

$$\left[\frac{V_{\rm c}(r)}{V_{200}}\right]^2 = \frac{1}{x}\frac{\ln(1+cx) - cx/(1+cx)}{\ln(1+c) - c/(1+c)}, \quad x = \frac{r}{r_{200}}$$

The NFW profile

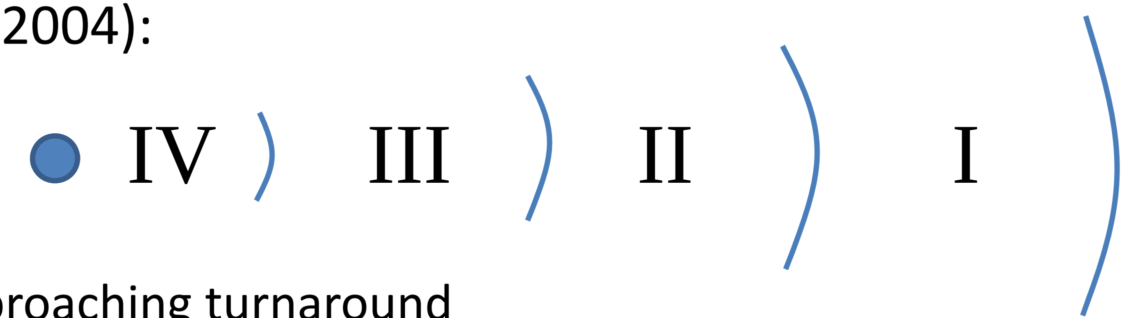
- Three papers by Navarro, Frenk & White:
  - 1995:
    - Resimulated clusters including gas; no recent major mergers; $\Omega=1$.
    - Profile does better than the cored β model in the centre and better than the steep Hernquist profile on the outside.
  - 1996:
    - Resimulated DM only haloes with a range of masses; $\Omega=1$; profile in its current form with relationship between concentration and characteristic density.
  - 1997:
    - DM haloes with different mass, initial power spectra, Ω and Λ; no recent major mergers.
    - Established universality of profile and related concentration to formation redshift.
    - Joint 18th most cited paper on ADS, with 2507 citations.

Explaining halo profiles dynamically

- Lynden-Bell (1967): violent relaxation.
- Gunn & Gott: secondary infall.
- Self-similar collapse: Fillmore & Goldreich (1984), Bertschinger (1985)… → $\rho \sim r^{-2.25}$
- NFW1997: "It appears that mergers and collisions during halo formation act as a 'relaxation' mechanism to produce an equilibrium that is largely independent of initial conditions."
- Syer & White (1998): Inner mass profile determined by tidal stripping: less dense satellites are disrupted and disperse throughout the halo, while more concentrated ones sink to the centre and steepen the inner cusp.
  - Later extended by Dekel et al. (2003).

Can hierarchical merging and violent relaxation provide an answer?

- Williams et al. (2004):



  - Region I: Approaching turnaround
  - Region II: Decoupled from Hubble flow
  - Region III: Shell-crossing region (c.f. Fillmore & Goldreich)
  - Region IV: Virialized region where tangential velocities reduce the slope
- Ascasibar et al. (2004): universality of the final profile comes from universality of the shape of the initial overdensities.
- Lu et al. (2006):
  - Fast accretion phase with velocity isotropization governs inner slope.
  - Slow accretion phase leads to steep outer slope.

Universal profile in other cosmologies

- A problem for any explanation of the universality of halo profiles which depends on merging is that the universality seems to extend to monolithic collapse scenarios.
- Huss, Jain & Steinmetz (1998): profiles independent of initial conditions and formation history.
- Wang & White (2008): NFW profiles fit the first generation of haloes in hot dark matter cosmologies, though the dependence of concentration on mass is reversed.
  - Same two phases of halo growth (rapid accretion and slow infall) as seen in CDM.
  - Other properties such as the phase space density profile (which is well fit by a power law) also seem to be very similar between the CDM and HDM case.

Maximum entropy method

- We seek the probability density function $f(t,\boldsymbol{x},\boldsymbol{v})$ which maximizes the Gibbs entropy,

$$S(t) = -\int f(t,\mathbf{x},\mathbf{v}) \ln f(t,\mathbf{x},\mathbf{v}) \mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{v}$$

  subject to all the relevant constraints.
- Can be justified by noting that this chooses the 'most uninformative distribution possible' (or the most probable/likely) given our constraints.
  - Choosing a distribution with higher entropy would violate the constraints.
  - Choosing a distribution with lower entropy is effectively using information we have chosen not to assume.

Constraints

- Conservation of mass: $M(r_{\mathrm{max}})=C_1$

$$M(r) = 4\pi \int_0^r \rho(r')r'^2 \mathrm{d}r'$$

- Conservation of energy: assume isotropy in velocity space, then $E_\mathrm{K}=C_2$ and $E_\mathrm{V}=C_3$ where

$$E_\mathrm{K} = 24\pi^2 G \int_0^\infty r^2 \mathrm{d}r \int_r^\infty \frac{\rho(r')}{r'^2} \mathrm{d}r' \int_0^{r'} \rho(r'')r''^2 \mathrm{d}r''$$

$$E_\mathrm{V} = -16\pi^2 G \int_0^\infty \rho(r) r \mathrm{d}r \int_0^r \rho(r')r'^2 \mathrm{d}r'$$

Constraints contd.

- So far all follows a standard development as can be found in Binney & Tremaine or in Dejonghe (1986). But here we also add a virialization constraint, $E_K=-\frac{1}{2}E_V$.
- Constants fixed by integrating an NFW profile(!) for a halo with c=17.5 and $M=2\cdot10^{12}M_{sun}$ at z=0 between r/r_s =0.2 and 25.

Parametrization

$$\rho(x) = f_1(x; a_0, a_1) + f_2(x; a_3, a_4) + \sum_{n=0}^{N} b_n x^n$$

$$-a_0 x - a_1 \ln(1 + e^x)$$

Reduces to the NFW profile for a_0=1 and a_1=2

$$a_3/(x_u^+ - x)^{a_4}$$

Fits the shape of the tails at large radius, which arise because the density must go to zero there

Just needs to be some reasonably complete set of functions. N=9 chosen for this paper

The minimization

- Uses Powell's method with an augmented Lagrangian (minimize, update parameters, reduce μ, minimize again, repeat…)

$$S_A(\mathbf{a},\lambda,\mu) = S(\mathbf{a}) - \sum_{i=1}^{3} \lambda_i h_i(\mathbf{a}) + \frac{1}{2\mu}\sum_{i=1}^{3} h_i^2(\mathbf{a})$$

Vector of parameters a_i , b_i

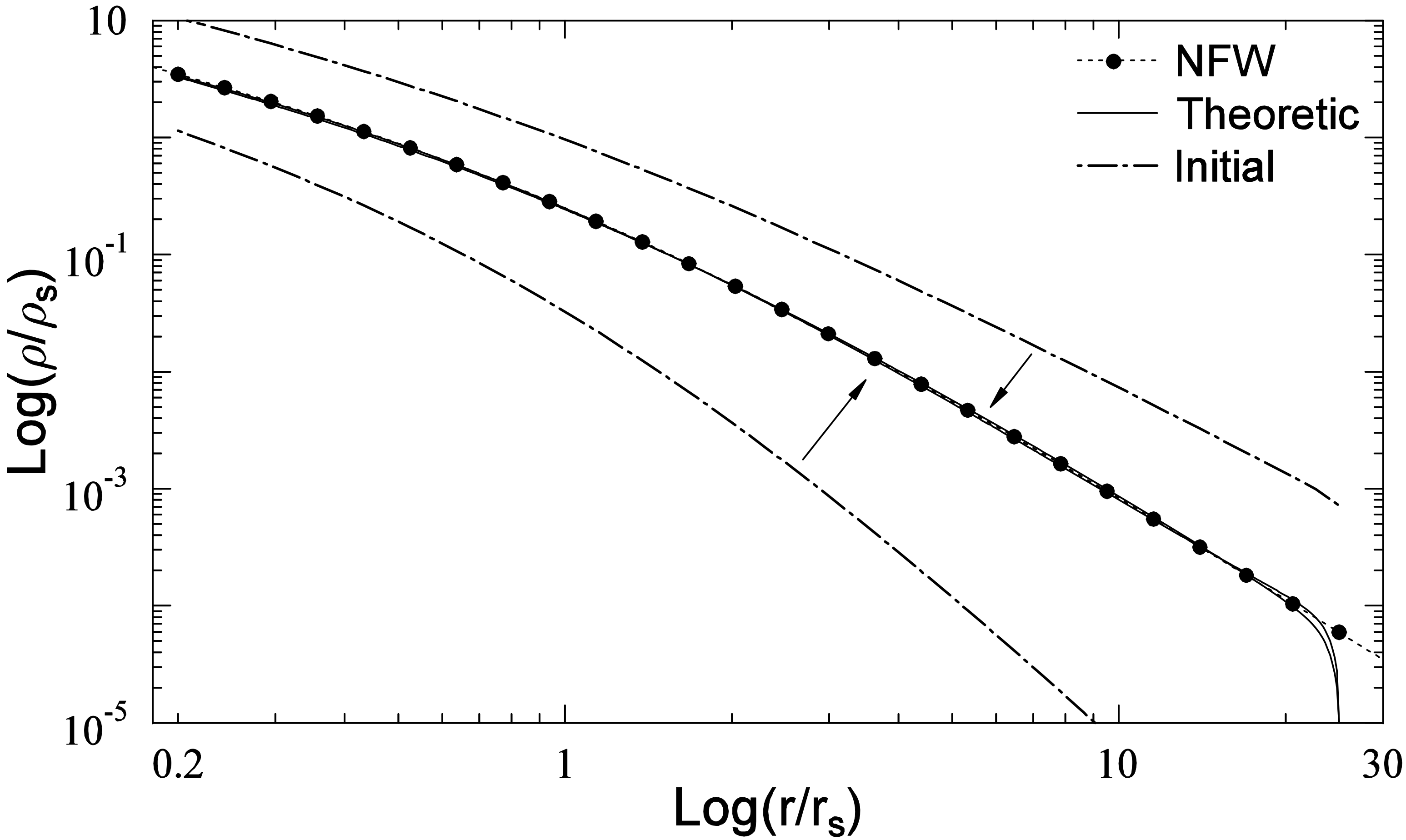
Objective function ([minus the] entropy)

Lagrange multipliers

'Penalty factor'

Constraints are given by h_i=0

Results



10
10^{-1}
10^{-3}
10^{-5}
$\mathrm{Log}(\rho/\rho_s)$
0.2
1
10
30
$\mathrm{Log}(r/r_s)$
NFW
Theoretic
Initial

Discussion & summary

- Need to be careful not to choose degenerate parameters or start with a solution which is too poor.
- No dynamical halo formation/evolution processes included, and universality seems to come for free since the profile comes from minimal assumptions…
- …so how does this square with studies claiming the profiles aren't universal ('We argue that the non-universality they found is the phenomenon coming from high-resolution simulations'!)?
- Nonetheless, getting the NFW profile from maximizing the entropy subject to constraints on mass and energy conservation and virialization is a nice result.

Shortcomings & questions

- Distribution function set to zero at the outer radius, which necessitates messing with the parametrization.
- Never shows the final values of the parameters: how close is the final distribution to the input NFW profile?
- There's been much discussion that the logarithmic slope doesn't really asymptote to a value of -1 in the centre: can this method address what we might expect? How about the slope in the outer parts?
- Is including angular momentum really 'unlikely to affect the conclusion significantly'?
- Effect of changes in mass/concentration/cosmology?
- Energy not currently independently derived from the mass.
- Uses collisionless Boltzmann equation; can the result be generalized for haloes with gas?